*Traces*

Why useful? Easier to visualize curves in $\mathbb{R}^2$ than visualizing surfaces in $\mathbb{R}^3$.

What are traces? (also called cross sections) (here: $k$ = real #)

Trace in $x = k$: curve obtained by intersecting/cutting the surface with the vertical plane $x = k$

Trace in $y = k$: __________________________ $y = k$

Trace in $z = k$: __________________________ horizontal plane $z = k$

Eg: show the 2 surfaces in p. 555 APEX

1) $z = y^2$

Trace in $x = k$: parabola

\[
\begin{cases}
    z = y^2 \\
    x = k
\end{cases}
\]

Trace in $y = k$: one line

\[
\begin{cases}
    y = k \\
    z = k^2 \\
    x = \text{anything}
\end{cases}
\]

Trace in $z = k$:

(*) say $k > 0$)
2) $x^2 + y^2 = 1$

Trace in $x = k$ :
(say $-1 < k < 1$)

Trace in $y = k$
(say $-1 < k < 1$)

Trace in $z = k$:

Cylinders: above surfaces are examples of cylinders

Roughly: cylinder = take the curve defined by an eq. of 2 variables then move across the axis of the remaining variable

Eq: $z = y^2$ (move across $x$-axis)

$x^2 + y^2 = 1$ (move across $z$-axis)
Conic sections & Quadric Surfaces

Conic sections

Roughly: curves in \( \mathbb{R}^2 \) given by polynomial equations of degree 2

3 types:

- Parabolas: \( y = x^2 \)  \( x = \frac{1}{2} y^2 \)  \( y = (x-1)^2 + 2 \)

\[ \begin{align*}
&\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\ quar......
Note: sometimes, shifting is needed to have the above form

\[ y = (x - 1)^2 + 2 \implies Y = X^2 \]

\[ \frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1 \implies \frac{X^2}{4} + \frac{Y^2}{3} = 1 \]

\[ \frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1 \implies \frac{Y^2}{9} - \frac{X^2}{4} = 1 \]

**Quadric surfaces**

Roughly: surfaces in \( \mathbb{R}^3 \) given by polynomial equations of degree 2

**Classification:**

1) **Ellipsoid**: all traces are ellipses

\[ \text{eq: } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

\[ \text{eg: show the ellipsoid } x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1 \text{ in p.562 APEX} \]

2) **Paraboloid**: 2 types of traces are parabolas
2a) Elliptic paraboloid: the remaining traces are ellipses

\[ Z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

\[ \Rightarrow \text{show p. 559 APEX} \]

2b) Hyperbolic paraboloid: the remaining traces are hyperbolas

\[ Z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \]

\[ \Rightarrow \text{show p. 561 APEX} \]

3) Cones, Hyperboloids of One Sheet, Hyperboloids of Two Sheets

2 types of traces are hyperbolas

\[ \Rightarrow \text{show p. 559, 560, 561 APEX} \]

**Example:**
Show how to roughly visualize the surface

\[ y = \frac{x^2}{4} + \frac{z^2}{16} \] (this is an elliptic paraboloid)

then show picture in p. 562
More pictures: see p. 584 in Stewart, do exercises 21-28 in Sec 12.6 of Stewart
Skip surface of revolution (you can read it yourselves)

§12.1 Multivariable Functions

2 variables:

- Domain $D$: subset of $\mathbb{R}^2$
- Every $(x, y)$ gives a value $f(x, y)$ in $D$

Graph of $f(x, y)$: all points $(x, y, z)$ such that $z = f(x, y)$ and $(x, y)$ in $D$

Draw picture:
eq: \( f(x, y) = \sqrt{9 - x^2 - y^2} \)

a) Find the domain \( D \)

b) \( f(1, 2) = ? \)

c) What is the graph?
Eq: previous eq \( f(x,y) = \sqrt{g-x^2-y^2} \)

Require that the domain must satisfy

\( x \geq 0, \ y \geq 0 \)

a) What is the domain?

b) Graph of \( f(x,y) \) under this more restricted domain?

(mention the word “first octant” here!)