§10.6 Planes

How to determine a plane in $\mathbb{R}^3$?

* Obvious answer: given 3 non-collinear points

* More convenient answer: given a point on the plane and a normal vector of the plane

(Here: normal vector of a plane is a vector that is orthogonal to the plane)

Problem: given $P(x_0,y_0,z_0)$ and $\vec{n} = \langle a, b, c \rangle$.

Describe the plane $S$ containing $P$ and orthogonal to $\vec{n}$.

Answer: describe random point $Q(x,y,z)$ on $S$.

Relation $\vec{PQ}$ and $\vec{n}$?

$\vec{PQ} \perp \vec{n}$ or $\vec{PQ} \cdot \vec{n} = 0$

Vector eq: $(\vec{OQ} - \vec{OP}) \cdot \vec{n} = 0$ or $\langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a, b, c \rangle = 0$
Equations of a plane:

- Scalar equation (also called "standard form" in the book):
  \[ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \]

- Linear equation:
  \[ ax + by + cz + d = 0 \]
  where \( d = -ax_0 - by_0 - cz_0 \)

  (note: slightly differs from the textbook: \( ax + by + cz = d \) where \( d = ax_0 + by_0 + cz_0 \))

Find eq of plane through 3 points

Eq: find eq for the plane through \( P(1,3,2), Q(3,-1,6), R(5,2,0) \)
(Hint: need \( \vec{n} \). Know: \( \vec{n} \perp \) any vector formed by \( \overrightarrow{PQ}, \overrightarrow{PR} \))

Answer:
\[
\begin{align*}
\overrightarrow{PQ} &= \langle 2, -4, 4 \rangle \\
\overrightarrow{PR} &= \langle 4, -1, -2 \rangle \\
\text{then } \vec{n} &= \langle 2, -4, 4 \rangle \\
\text{Answer: } &
\end{align*}
\]

Find eq for the line that is the intersection of 2 planes
Method: pick a point \( P \) in the intersection, then

**Method 1:** find \( \mathbf{d} \) parallel to the intersection line

Know: \( \mathbf{d} \perp \mathbf{n}_1 \) and \( \mathbf{d} \perp \mathbf{n}_2 \) \( \Rightarrow \) choose \( \mathbf{d} = \mathbf{n}_1 \times \mathbf{n}_2 \)

**Method 2:** pick another point \( Q \) \( \Rightarrow \) line through \( P \) \& \( Q \)

**Eq:** Find symmetric eq for the intersection of the planes

\[
S_1: \quad x + y + z = 1
\]

and \( S_2: \quad x - 2y + 3z = 1 \)

Answer: pick \( P \) in the intersection, solve \( \begin{cases} x + y + z = 1 \\ x - 2y + 3z = 1 \end{cases} \)

We only need one point \( P \), set \( x = 1 \) \& solve for \((y,z)\) (you can set \( x = 100 \) \& solve for \((y,z)\) or set \( y = 0 \) \& solve for \((x,z)\), etc.)

\[
\begin{align*}
1 + y + z &= 1 \\
1 - 2y + 3z &= 1
\end{align*}
\]

\( \Rightarrow \) \( y = z = 0 \)

\( P(\quad) \)

**Method 1:** \( \mathbf{n}_1 = \langle \quad \rangle \), \( \mathbf{n}_2 = \langle \quad \rangle \)

\( \mathbf{d} = \)

Answer:

**Method 2:** pick another \( Q \): set \( x = 0 \) \& solve for \((y,z)\):

\[
\begin{align*}
y + z &= 1 \\
-2y + 3z &= 1
\end{align*}
\]

\( \Rightarrow \) \( y = 1 - z \), \( z = \quad \), \( y = \quad \)

\( Q(\quad) \)
take \( \vec{d} = \overrightarrow{PA} < \) 

answer:

* Distance from a point to a plane (a distance between 2 parallel planes)

Find distance from \( C \) to the plane \( S \)

Here distance = length of \( \overrightarrow{DC} \)

\[
\overrightarrow{DC} = \text{proj}_n \overrightarrow{PC}
\]

\[
\Rightarrow \text{distance} = \text{length of } \text{proj}_n \overrightarrow{PC} = \]

(thus is Key Idea 5.1 in p.620)

A useful formula (that the book doesn't give you):

If \( S \) is given by the eq \( ax + by + cz + d = 0 \) and \( C(x_1, y_1, z_1) \) then the distance from \( C \) to \( S \) is
Eg: Given 2 planes:

\[ S_1: \quad 10x + 2y - 2z = 5 \]
\[ S_2: \quad 5x + y - z = 1 \]

a) Why are they parallel?
b) Distance between them?

answer: a)

b) Pick any point \( C \) on \( S_1 \), find distance from \( C \) to \( S_2 \).
To pick \( C \), need \( 10x + 2y - 2z = 5 \).

* Angles: *(not in textbook, could be asked in Webwork, exam, ...)*

Rule: - use acute angles (i.e. \( \leq 90^\circ \))
    - Draw picture, use normal vectors, and see.

* Angle between 2 planes:
In these pictures, the angle between 2 planes should be $\theta_1$.

It turns out $\theta_1 = \theta_2$.

And $\theta_2$ is "cut out" by $\vec{n}_1$ and $\vec{n}_2$.

The angle between 2 planes is the **acute** angle formed by 2 normal vectors.

**Eq:** Find the angle between the planes $x + 2y + 3z = 0$ and $3x - 4y = 1$

**Ans:** $\vec{n}_1 = <a, b, c>$, $\vec{n}_2 = <d, e, f>$

$\theta = \text{angle between } \vec{n}_1 \text{ and } \vec{n}_2$, $\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{||n_1|| ||n_2||}$

So $\theta = \text{ }$

**Warning:** since we want an acute angle, the final answer is [ ] (which is also )
* Angle between a line & a plane

Find \( \theta_2 = \text{acute angle between L and } \vec{n} \)

answer: angle between L and S is \( \theta_1 = \frac{\pi}{2} - \theta_2 \)

Eq: find the angle between the plane \( 3x - 4y = 1 \) and the line \( x - 1 = \frac{y}{2} = z \)

answer: first, find the acute angle cut out by L and \( \vec{n} = \langle 3, -4, 0 \rangle \). A vector parallel L is \( \vec{d} = \langle \ldots \rangle \)

\( \theta = \text{angle between } \vec{n} \text{ and } \vec{d} \)

\( \cos \theta = \)

the acute angle is

answer: \( \frac{\pi}{2} - \)
Cylinders and Quadric Surfaces

(Skip surfaces of revolution; Sec 10.1 in Apex book)
Sec 12.6 in Stewart

Warm-up: In $\mathbb{R}^3$

a) Does the eq. $x^2 + y^2 = 1$ represent a circle?

b) What is it & sketch it?

Answer:

Our goal:

- Learn how to sketch cylinders
- Sketch surfaces using traces in $x$, $y$, $z$
- Recall conic sections: ellipses, parabolas, hyperbolas
- Sketch & visualize ellipsoids and simple elliptic paraboloids
  such as $y = 2x^2 + z^2$, etc.