Conclusion. (The book uses \( \overrightarrow{OP} \) for \( \overrightarrow{OP} \) and \( \overrightarrow{OQ} \) for \( \overrightarrow{OQ} \))

1) Vector equation for \( L \):
\[
\vec{L}(t) = \overrightarrow{P} + t \overrightarrow{d}
\]
for \( t \) in \( \mathbb{R} \)

2) Parametric equation:
\( L \) consists of points \( Q(x,y,z) \) such that
\[
x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct
\]
for \( t \) in \( \mathbb{R} \)

3) Symmetric equation: ("solve for \( t \")
\( L \) consists of points \( Q(x,y,z) \) such that
\[
\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}
\]

Remark for symmetric eq: if, say, \( a = 0 \)
remove \( \frac{x-x_0}{a} \), add the equation \( x = x_0 \)

Hence
\[
\begin{cases}
\frac{y-y_0}{b} = \frac{z-z_0}{c} \\
x = x_0
\end{cases}
\]

Q: What's the symmetric eq if \( a = b = 0 \)?
\[ x = x_0 \quad \text{and} \quad y = y_0 \quad (\text{so} \ z \ \text{is any real number}) \]

(in this case, \( \vec{d} = \langle 0, 0, c \rangle \), hence the line is parallel to \( z \)-axis)

Eq: quickly pick out a point \( P \) on the line and a vector \( \vec{d} \) parallel to the line.

a) line \( L \) given by \[ \begin{cases} x = 5 \\ y = \pi t \\ z = 1 + e^t \end{cases}, \quad t \in \mathbb{R} \]

Choice of \( P \) and \( \vec{d} \)?

Answer: \( \vec{d} = \langle 0, \pi, e \rangle \), \( P(5, 0, 1) \)

b) line \( L \) given by \[ \frac{x-1}{2} = \frac{3y-4}{5} = \frac{-6z-7}{8} \]

Choice of \( P \) and \( \vec{d} \)?

Answer: rewrite it to \[ \frac{x-1}{2} = \frac{y-\frac{4}{3}}{5/3} = \frac{z + \frac{7}{6}}{-8/6} \rightarrow \begin{pmatrix} \text{Wrong if you} \\ \text{think} \langle 2, 5, 8 \rangle \end{pmatrix} \] \( \Rightarrow \vec{d} = \langle 2, \frac{5}{3}, -\frac{8}{6} \rangle \), \( P(1, \frac{4}{3}, -\frac{7}{6}) \)

Eq: different parametric equations can represent the same line.

Explain why these lines \( L_1 \) and \( L_2 \) are the same.

\[ \begin{align*}
L_1: & \quad \begin{cases} x = 1 + 2t \\ y = 3 + 4t, \ t \in \mathbb{R} \\ z = 5 + 6t \end{cases} \\
L_2: & \quad \begin{cases} x = t \\ y = 1 + 2t, \ t \in \mathbb{R} \\ z = 2 + 3t \end{cases}
\]
Eq: determine eq for line with 2 given points.

Find symmetric & parametric eq of the line through $A(2,4,2)$ and $B(3,7,-2)$

Eq: Find the intersection of the above line with the $xz$-plane

Answer:
In $\mathbb{R}^2$, we also have similar formulas

\[ \mathbf{P}(x_0, y_0) \text{ and } \mathbf{d} = \langle a, b \rangle \]

**Parametric**

\[
\begin{align*}
    x &= x_0 + at, \\ y &= y_0 + bt, \\
    t &\text{ in } \mathbb{R}
\end{align*}
\]

**Symmetric**

\[
\frac{x - x_0}{a} = \frac{y - y_0}{b}
\]

(as before, if $a = 0$, you have the vertical line $x = x_0$)

* Compare 2 lines in $\mathbb{R}^3$: 4 possibilities for $L_1$ & $L_2$

1) **Parallel**: check if $\mathbf{d}_1 \parallel \mathbf{d}_2$

1a) Parallel & different

1b) 2 lines are the same

How? Once you know they are parallel: pick $P$ on $L_1$, check if $P$ is on $L_2$

No $\Rightarrow$ different lines

Yes $\Rightarrow$ same line

**Eq:**

\[ L_1: \begin{cases} \mathcal{H} = 1 + t \\ y = 2 + t, \ t \in \mathbb{R} \\ z = 3 + t \end{cases} \]

\[ L_2: \begin{cases} x = -1 - t \\ y = -2 - t, \ t \in \mathbb{R} \\ z = -3 - t \end{cases} \]

Why parallel?

Why not the same line?
2) Not parallel

2a) Not parallel and intersect

2b) Not parallel and do not intersect

in this case, $L_1$ and $L_2$ are called skew lines.

How to know? First, make sure they are not parallel, then find the point of intersection if there is such a point $\Rightarrow 2a$ if there isn't $\Rightarrow 2b$

How to find point of intersection? Solve a linear system of 3 equations in 2 variables: pick 2 equations you can solve, then plug-in the solution to the third to check.

Eq. lines

$L_1: \quad x - 1 = \frac{y - 1}{2} = \frac{z - 1}{3}$

$L_2: \quad x - 2 = \frac{y - 3}{3} = \frac{z}{5}$

Are they skew lines?

Answer:

• Not parallel since:

• Rewrite into parametric equations:
\( L_1: \begin{cases} x = 1 + t \\ y = 1 + 2t, \ t \in \mathbb{R} \\ z = 1 + 3t \end{cases} \)

\( L_2: \begin{cases} x = 2 + t \\ y = 3 + 3t, \ t \in \mathbb{R} \\ z = 5t \end{cases} \)

To find the point of intersection, solve:

\[
\begin{aligned}
1 + t &= 2 + s \\
1 + 2t &= 3 + 3s \\
1 + 3t &= 5s
\end{aligned}
\]

!!! You should understand why we use a different parameter "s" here. When we think of each line separately, the dummy variable \( t \) represents a random number. When we study these 2 lines at the same time & find the point of intersection, this point could come from "one \( t \)" for \( L_1 \) and "another \( t \)" for \( L_2 \).
Distance to a line

1. From a point to a line
   
   - Pick any A on L
   
   - Previous method:
     
     \[ \overrightarrow{AD} = \text{proj}_d \overrightarrow{AC} \]
     
     then find \[ \overrightarrow{DC} = \overrightarrow{AC} - \overrightarrow{AD} \] take \( \parallel \overrightarrow{DC} \parallel \)
   
   - Another method:
     
     \[
     \text{take height of parallelogram} \quad \frac{\text{area}}{\text{base}} = \]

   Example: distance from Q(1,1,3) to line \( \ell(t) = (1,1,1) + t(2,3,1) \)

   Answer: pick the point \( A(\quad) \) on the line, \( \overrightarrow{d} = \langle \quad \rangle \)

   \[ \overrightarrow{AQ} = \]
   
   \[ \overrightarrow{AQ} \times \overrightarrow{d} = \]

   Answer: \[ \frac{\parallel \overrightarrow{AQ} \times \overrightarrow{d} \parallel}{\parallel \overrightarrow{d} \parallel} = \]

2. Between 2 parallel lines
   
   Easy: pick any C on one line, find distance to other using (1).
(3) Hardest: distance between 2 skew lines

- Find p.610 & show picture
- Key idea 50 in p.614
- Here's the explanation:

  - distance = shortest length \|Q_1Q_2\|
    
    with Q_1 in L_1, Q_2 in L_2
    
    - turns out: Q_1Q_2 \perp both L_1 and L_2
      
      \[ \Rightarrow \text{find } \vec{c} = \vec{d}_1 \times \vec{d}_2 \]

- Pick random P_1 on L_1 and P_2 on L_2
  
  \[ \Rightarrow \text{relation between } \vec{Q_1Q_2} \times \vec{P_1P_2} \]

- Length of projection: recall

  \[ \text{proj}_\vec{v} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \left( \frac{\vec{v}}{\|\vec{v}\|} \right) \]

  \[ \Rightarrow \text{length of projection } = \frac{|\vec{u} \cdot \vec{v}|}{\|\vec{v}\|} \]

Formula: Pick P_1 on L_1, P_2 on L_2, find \[ \vec{c} = \vec{d}_1 \times \vec{d}_2 \]

distance =
Eq: previously, the lines
\[ L_1: \ x - 1 = \frac{y - 1}{2} = \frac{z - 1}{3} \quad \text{and} \quad L_2: \ x - 2 = \frac{y - 3}{3} = \frac{z}{5} \]
are skew lines. Find the distance between them?

Ans: pick \( P_1(\quad) \), \( P_2(\quad) \)
\[ \Rightarrow \vec{P_1P_2} = \]
\[ d_1 = \langle \quad \rangle, d_2 = \langle \quad \rangle \]
\[ \vec{c} = d_1 \times d_2 = \]

\[ \text{distance} = \text{length of proj}_{\vec{c}} \vec{P_1P_2} = \frac{|\vec{P_1P_2} \cdot \vec{c}|}{||\vec{c}||} = \]

\underline{Line segment (read thin at home, not really important)}

\[ P_0(x_0, y_0, z_0) \]
\[ P_1(x_1, y_1, z_1) \]

describe all points \( Q(x, y, z) \) in the line segment \( P_0P_1 \):
\[ \overrightarrow{OQ} = \overrightarrow{OP_0} + \overrightarrow{P_0Q} \]
\[ = \overrightarrow{OP_0} + t \overrightarrow{P_0P_1} \]
\( (\text{where } 0 \leq t \leq 1 \text{ since } Q \text{ between } P_0 \& P_1) \)
\[ = \overrightarrow{OP_0} + t(\overrightarrow{OP_1} - \overrightarrow{OP_0}) \]
\[ = (1-t)\overrightarrow{OP_0} + t\overrightarrow{OP_1} \]

Parametric eq:
\[
\begin{align*}
    x &= (1-t)x_0 + tx_1 \\
    y &= (1-t)y_0 + ty_1 \quad \text{where } 0 \leq t \leq 1 \\
    z &= (1-t)z_0 + t z_1
\end{align*}
\]
§10.6 Planes

How to determine a plane in $\mathbb{R}^3$?

- Obvious answer: given 3 non-collinear points
- More convenient answer: given a point on the plane and a normal vector of the plane (i.e., normal vector of a plane is a vector)

Problem: given $P(x_0, y_0, z_0)$ and $\vec{n} = \langle a, b, c \rangle$.

Describe the plane $S$ containing $P$ and orthogonal to $\vec{n}$.

Answer: describe random point $Q(x, y, z)$ on $S$.

Relation $\vec{PQ}$ and $\vec{n}$?

Vector eq: \[
(\vec{OQ} - \vec{OP}) \cdot \vec{n} = \langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a, b, c \rangle = \]