Eg 1 in p. 983: the textbook does something clever, I'll show you why the above trick works too (so here's another way to do this eg different from the book):

Rectangular box without lid made from 12 m² cardboard

Maximize volume?

Solution: set up the notation: \( x = \) length, \( y = \) width, \( z = \) height

Maximize \( f(x, y, z) = xyz \)

under constraint \( xy + 2xz + 2yz = 12 \)

(only 1 \( xy \) since no lid)

\( g(x, y, z) \)

Solve the system:

\[
\begin{align*}
yz &= \lambda (y + 2z) \\
xz &= \lambda (x + 2z) \\
xy &= \lambda (2x + 2y) \\
xy + 2xz + 2yz &= 12
\end{align*}
\]

Solve for \( \lambda \):

\[
\begin{align*}
\lambda &= \frac{yz}{y + 2z} \\
\lambda &= \frac{xz}{x + 2z} \\
\lambda &= \frac{xy}{2x + 2y}
\end{align*}
\]

(Ok to divide since \( x, y, z = \) length, width, height \( > 0 \)
so all the above denominators \( > 0 \))

Now equate the above RHS of \( \lambda \) to get relations among \( x, y, z \):
\[
\frac{y}{x^2 + 2z} = \frac{x}{x^2 + 2z}
\]

\[xy + 2yz = xy + 2xz\]

\[2y^2 = 2x^2\]

\[\Rightarrow y = x\]

So \(x = y = 2z\)

Plug-in \(x = y\) and \(z = \frac{y}{2}\) (well, you can plug in \(x = 2z\) & \(y = 2z\) too) into the last equation (the constraint), get:

\[y^2 + y^2 + y^2 = 12\]

\[3y^2 = 12\]

\[y = 2\] (note that \(y > 0\))

\[\Rightarrow x = 2\] and \(z = 1\)

Only solution \((x, y, z) = (2, 2, 1)\)

So max volume = \(\int (2, 2, 1) = 4\)

---

Another eg:

E: ellipsoid given by \(x^2 + 2y^2 + 4z^2 = 28\) (problem in the 2nd midterm in 2014)

F: plane given by \(x + y + z = 9\)

Find a point on E that is closest to F and a point on E that is farthest from F.

Calculate the corresponding min & max distance
(given point \((x, y, z)\), need to max/min distance = 

for the point to be on the ellipsoid, you need the constraint:

\[
\text{Max/min} \quad \text{or} \quad \text{instead,}
\]

then modify the final answer accordingly.

We will max/min \(f(x, y, z) = \) under the constraint. Solve:

\[
\begin{align*}
\text{Warning: if you lazily write} & \\
(\pm 4, \pm 2, \pm 1), \text{you actually} & \\
\text{mean 8 solutions!!!}
\end{align*}
\]
Lagrange multipliers with 2 constraints

\[ g(x, y, z) = k \]
&
\[ h(x, y, z) = c \]

**Method:** solve the system

\[
\begin{align*}
\nabla f &= \lambda \nabla g + \mu \nabla h \\
g(x, y, z) &= k \\
h(x, y, z) &= c
\end{align*}
\]

in variables \( x, y, z, \lambda, \mu \)

(here it means \( \nabla f \) is a "linear combination" of \( \nabla g \) and \( \nabla h \))

- How to solve the system: this is harder than the previous system, you can try to solve for \( \lambda \) and \( \mu \) as before, then get out relation in \( x, y, z \) as before.

- In the final: for Lagrange multipliers with 2 constraints, we'll only require you to set up the above system, you don't need to solve such a system.

- Eg: read example 5 in p. 986 Stewart (do it yourself)
- Eg: explain Example 14.8.2 in Secondary Text #1 p.381