Overview:

- We will solve 3 types of optimization problems:
  1. Find & classify critical points
  2. Find max/min over a "closed & bounded" domain
  3. Find max/min under a constraint using the method of Lagrange multipliers.

2 comments

* Once you learn everything, you'll see that (2) is sort of a combination of (1) & (3)

* Textbooks:
  - Stewart treats all 3 problems (Sec 14.7 & 14.8)
  - APEX only treats (1) & (2), unfortunately
  → refer to Secondary Textbook #1 (Community Calculus) for (3).
* Understand the difference

1) global/absolute max:

vs

local/relative max:

2) global/absolute min

vs

local/relative min

* Show picture in p.970 Stewart

* Critical points:
  
  - Recall previous calculus:
    \( f(x) \): function in 1 variable. Local max/min only possible at critical points
  
  either

  or
Back to Math 200: $f(x,y)$ function in 2 variables $(a,b)$ is a critical point if either at least one of $f_x(a,b)$ & $f_y(a,b)$:

\[ f_x(a,b) \text{ or } f_y(a,b) \text{ exist and } \]

In our class, $f$ (almost always) has a nice formula and $f_x \text{ and } f_y$ exist $\Rightarrow$ only care about the boxed condition.

Repeat: (see Theorem 2 in p.970 Stewart)

$f$ has local max/min at $(a,b)$ and $f_x(a,b), f_y(a,b)$ exist then $f_x(a,b) = f_y(a,b) =$

Some geometric intuition: graph $Z = f(x,y)$ at local max/min $(a,b)$:

. What does tangent plane look like?

. Why $\nabla f(a,b) = \vec{0}$, what if $\nabla f(a,b) \neq \vec{0}$?

Max $D_{\vec{u}} f(a,b) = \nabla f(a,b) > 0 \Rightarrow$ increasing $\Rightarrow$ can't be max

Min $D_{\vec{u}} f(a,b) = \nabla f(a,b) < 0 \Rightarrow f$ decreasing $\Rightarrow$ can't be min
Eq: \( f(x,y) = x^2 + y^2 - 2x - 6y + 14 \)

a) Find the points \((a, b)\) where \(f\) might have local max or min.

b) Soon you'll be able to explain why the point in part (a) gives you a local min. Can you explain why it's even a global min (this usually requires some effort)?

Solution

Recall (previous calculus) \( f(x): 1\)-var. Once we know \( f'(a)=0\), how to determine if \( a \) is a local min/max?

2nd derivative test (1-variable)

If \( f''(a) > 0 \):

\( f''(a) < 0 \):

\( f''(a) = 0 \):
Second Derivative Test \((\text{Stewart p. 971})\): \(f(x,y)\): \(2\)-var

Know \(f_x(a,b) = f_y(a,b) = 0\). let

\[ D = \]

If
\[ D > 0 \text{ and } D > 0 \text{ and } D < 0 \]

(show picture of saddle point in p.971) of Stewart

Remark: if \(D = 0\): the test is inconclusive, \((a,b)\) could give you anything

\[ \xrightarrow{\text{local max}} \leftarrow \text{local min} \quad \xrightarrow{\text{saddle}} \]

Typical/simplist eg: classify the critical points of

a) \(f(x,y) = x^2 + y^2\); \hspace{1cm} b) \(f(x,y) = -x^2 - y^2\)

c) \(f(x,y) = x^2 - y^2\)
Eq: find x and classify the critical points of \\
\[ f(x,y) = x^4 + y^4 - 4xy + 1 \]

\underline{Remark}: Done with local max/min. How to find global max/min? \\
This is tricky in general. Use algebra/graph/intuition... or \\
simply assume global max/min exists (since you are asked for it) then take \\
\[
\begin{align*}
\max \text{ (all local)} &= \text{ global max} \\
\min \text{ (all local)} &= \text{ global min}
\end{align*}
\]
Eg: Previous \( f(x,y) = x^4 + y^4 - 4xy + 1 \). Try to find global max \& global min?

Eg: (exercise 42 p. 979) Find points on the surface \( y^2 = 9 + xz \) that is closest to the origin \& find this smallest distance. Solution: need to find min of \( \sqrt{x^2 + y^2 + z^2} \) under "constraint" \( y^2 = 9 + xz \).

2 methods: Method 1: solve for 1 variable \( \Rightarrow \) plug-in get a function in 2 vars without constraint.

Method 2: Lagrange multiplier in §14.8 Stewart (secondary textbook) #1.

Let's do method 1: find min of \( f(x,y,z) = x^2 + y^2 + z^2 \) under condition:

\( y^2 = 9 + xz \)

(slightly nicer than \( \sqrt{x^2 + y^2 + z^2} \); at the end, take \( \sqrt{ } \) of your answer.)