- Review example from last time on "hill $z = 10000-..-..")

- Next topic: Directional Tangent Lines

  Stewart §14.6
  
  \begin{array}{l}
  \{ \text{Tangent Planes of Level Surfaces} \} \\
  \{ \text{Normal Lines} \} \\
  \end{array} \\
  \text{APEX §12.7}

\* Directional tangent lines (APEX p.730-732) (not really important, do this quickly)

- Graph $z = f(x,y)$ at $(x_0, y_0, z_0)$.

Previously, in the note Filled-Sep28

\[
\begin{align*}
\frac{\partial f}{\partial x}(x_0, y_0) &= \text{slope of the tangent of the trace in } y = y_0 \\
\frac{\partial f}{\partial y}(x_0, y_0) &= x = x_0 \\
\end{align*}
\]

\Rightarrow \text{Show Figure 12.20 in p.730 APEX, note the 3 red lines}

So the tangent of the trace in $y = y_0$ has equation:

\[
\begin{align*}
\begin{cases}
  x = x_0 + t \\
  y = y_0 \\
  z = z_0 + f_x(x_0, y_0)t \\
\end{cases}
\Rightarrow \text{parallel } \langle 1, 0, f_x(x_0, y_0) \rangle \\
\text{think of this as tangent line in the } x\text{-direction (direction of } \langle 1, 0 \rangle) \\
\end{align*}
\]

The tangent of the trace in $x = x_0$ has eq:

\[
\begin{align*}
\begin{cases}
  x = x_0 \\
  y = y_0 + t \\
  z = z_0 + f_y(x_0, y_0)t \\
\end{cases}
\Rightarrow \text{parallel } \langle 0, 1, f_y(x_0, y_0) \rangle \\
\text{think of this as tangent line in the } y\text{-direction (direction of } \langle 0, 1 \rangle) \\
\end{align*}
\]
More generally: \( \vec{u} = \langle u_1, u_2 \rangle \) is a unit vector, the directional tangent line is the line through \((x_0, y_0, z_0)\) and parallel to

Eq: Find the line (eg. find parametric eq) tangent to the surface \( z = \sin x \cos y \) at \( \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \) in the direction of \( <-1,1> \)

* Next topic is far more important: tangent planes of level surfaces

Slogan: gradient vector \( \perp \) tangent plane of level surface
Tangent Plane (2nd formula applied for level surface)

\[ F(x, y, z) : \text{function in 3 variables} \]

(level) surface \( S \) given by \( F(x, y, z) = k \) \( (k: \text{some constant}) \)

At the point \((x_0, y_0, z_0)\), the tangent plane has

- Normal vector =
- Equation:

Normal line of a surface = line orthogonal to tangent plane

(so it's parallel to normal vector of tangent plane)

Eq: find eq of the tangent plane & symmetric eq of the normal line of the ellipsoid

\[ \frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \text{ at the point } (-2, 1, -3) \]

Solution:
with \( \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \)

So normal \( \langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \rangle = \langle -\frac{F_x}{F_z}, -\frac{F_y}{F_z}, -1 \rangle \)

parallel to \( \langle F_x, F_y, F_z \rangle = \nabla F \Rightarrow \) recover 2nd formula

An example:

\* Solve Problem 1 in MT2 Section 105 (in 2015)

1a) Here's a \text{WRONG} solution given by many students:

\[
\begin{align*}
\text{Given: } yz + x \ln y - z^2 &= 1 \\
\Rightarrow \ f_x &= \ln y, \quad f_y = Z + \frac{x}{y}
\end{align*}
\]

\[
\begin{align*}
Z &= Z_0 + f_x(x_0, y_0)(x-x_0) + f_y(x_0, y_0)(y-y_0) \\
Z - z &= \ln e \cdot (x-1) + (e + \frac{1}{e})(y-e) \\
\text{Answer: } Z - z &= x-1 + (e + \frac{1}{e})(y-e)
\end{align*}
\]

And here's the correct solution

1a) tangent plane of \( yz + x \ln y = z^2 + 1 \) at \( P(1, e, e) \)
Important remark: you must not mix up the 2 formulas for tangent plane.

1st formula: surface \( z = f(x, y) \) (graph of \( f(x, y) \))
\[
z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad \text{(normal \( \langle f_x, f_y, -1 \rangle \))}
\]

2nd formula: level surface \( F(x, y, z) = k \)
\[
F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0
\]
(normal \( \nabla F \))

Remark #1: 2nd formula is more general.

If you have graph \( z = f(x, y) \),
you can rewrite this as \( f(x, y) - z = 0 \)
\[
F(x, y, z) = 0
\]

\( \text{normal} = \nabla F = \langle f_x, f_y, -1 \rangle \) \( \Rightarrow \) recover 1st formula.

Remark #2: well, if you think a bit more, 1st formula also gives 2nd formula.

level surface \( F(x, y, z) = k \)
\( \Rightarrow \) this defines implicitly \( z = f(x, y) \)
1b) \( z \) defined implicitly by \( yz + x \ln y = z^2 + 1 \). Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) at \( P(1, e, e) \).

1c) \( z \) implicit as in 1b), use linear approx. to approx. the value of \( z \) at \( x = 0.99 \) and \( y = e + 0.01 \).
**Tangent lines of level curves**

- Sort of important, but I'll let you read it at home if not enough time)

  See the picture in p. 966 in Stewart

- Quick discussion on lines in \( \mathbb{R}^2 \): we have studied parametric & symmetric eq of lines in both \( \mathbb{R}^3 \) & \( \mathbb{R}^2 \).

  For lines in \( \mathbb{R}^2 \) (not true in \( \mathbb{R}^3 \)), there's another way as you have seen in Calculus 1:

  Lines in \( \mathbb{R}^2 \) can be given by

  \[
  ax + by + c = 0
  \]

  (\( \star \))

  or \( a(x-x_o) + b(y-y_o) = 0 \) with \((x_o, y_o)\) is a point on your line & \( c = -ax_o - by_o \).

  So what is \( \langle a, b \rangle \) here? As you might have guessed, \( \langle a, b \rangle \) is a vector orthogonal to your line (eg: line \( y = 2x + 3 \) then the vector \( \langle 2, -1 \rangle \) is orthogonal to your line)

**Tangent line of level curves**

Level curve \( f(x, y) = k \), find an equation for tangent at \((x_o, y_o)\)?

\[
\nabla f(x_o, y_o) = \langle f_x(x_o, y_o), f_y(x_o, y_o) \rangle
\]

is orthogonal to your tangent line

**Tangent line eq:**

\[
f_x(x_o, y_o)(x-x_o) + f_y(x_o, y_o)(y-y_o) = 0
\]
Eq: In $\mathbb{R}^2$, find parametric eq for the tangent line of the circle $x^2 + y^2 = 1$ at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Solution: $f(x, y) = x^2 + y^2$, above circle is just level curve (at level $k = 1$), $f_x = 2x$, $f_y = 2y$. Then $f_x \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = 1$, $f_y \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \sqrt{3}$.

In $\mathbb{R}^2$, tangent line of the circle at $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is $x - \frac{1}{2} + \sqrt{3}(y - \frac{\sqrt{3}}{2}) = 0$.

Q1: how to write a parametric eq from this?

$\Rightarrow$ Put $x = t$, solve for $y$:

$t - \frac{1}{2} + \sqrt{3}(y - \frac{\sqrt{3}}{2}) = 0$

$y = -\frac{1}{\sqrt{3}}t + \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2}$

Answer:

$$\begin{cases} x = t \\ y = -\frac{1}{\sqrt{3}}t + \frac{1}{2\sqrt{3}} + \frac{\sqrt{3}}{2} \end{cases}$$

Q2: how about symmetric

From $x - \frac{1}{2} + \sqrt{3}(y - \frac{\sqrt{3}}{2}) = 0$, get

$$\frac{x - \frac{1}{2}}{1} = \frac{y - \frac{\sqrt{3}}{2}}{-\frac{1}{\sqrt{3}}}$$