Applications of Triple Integrals

* Volume: \( \text{Vol}(E) = \iiint_E 1 \, dV \)

* Moments & center of mass for a 3-d object:
  (also called centroid)

Solid object occupies region \( E \) with density function \( \rho(x,y,z) \). Then

- total mass \( m = \iiint_E \rho(x,y,z) \, dV \)
- Moment about the \( yz \)-plane:
  Now, take directed distance from the \( yz \)-plane = \( x \)-coord.

Formula:
\[
M_{yz} = \iiint_E x \rho(x,y,z) \, dV
\]

Similarly: moment about the \( xz \)-plane
\[
M_{xz} = \iiint_E y \rho(x,y,z) \, dV
\]

Moment about the \( xy \)-plane
\[
M_{xy} = \iiint_E z \rho(x,y,z) \, dV
\]
The center of mass (also called centroid) of \( E \):

\[
\overline{x} = \frac{M_{yz}}{m} = \frac{1}{m} \iiint_E x \rho(x,y,z) \, dV
\]

\[
\overline{y} = \frac{M_{xz}}{m} = \frac{1}{m} \iiint_E y \rho(x,y,z) \, dV
\]

\[
\overline{z} = \frac{M_{xy}}{m} = \frac{1}{m} \iiint_E z \rho(x,y,z) \, dV
\]

Eg: Find the center of mass of the solid of constant density bounded by: the cylinder \( x = y^2 \) & the planes \( x = z, \ z = 0, \ x = 1 \)

(Due to time constraint, just [find \( m \) and \( \overline{x} \]. You are)

(referred to Eg 6 p. 1048 for \( \overline{y} \) & \( \overline{z} \) Stewart)

Sol: describe the solid picture

then take the “slanted plane” \( x = z \) as the top & the plane \( z = 0 \) as the bottom.
Description: "Inside" the cylinder $x = y^2$  
\[ \Rightarrow x \geq y^2 \]  
"Behind" the plane $x = 1$  \[ \Rightarrow x \leq 1 \]  
"Below" the plane $x = z$  \[ \Rightarrow x \geq z \]  
"Above" the plane $z = 0$  \[ \Rightarrow z \geq 0 \]
1) Further comments on $\text{Vol} = \iiint 1 dV$:

* Solid: above the domain $D$ in the $xy$-plane and below the graph $z = f(x,y)$.

Last week, I kept saying: $\text{Vol} = \iiint_D f(x,y) dA$, but now we have $\text{Vol} = \iiint_{\text{solid}} 1 dV$. What's wrong? [Nothing!]

Start with $\iiint_{\text{solid}} 1 dV$.

Describe the solid: $0 \leq z \leq f(x,y)$ (above $xy$-plane), $z = f(x,y)$ (below graph) domain $D$ for $(x,y)$.

So, $\iiint_{\text{solid}} 1 dV = \iiint_{D} \left( \int_{0}^{f(x,y)} 1 \, dz \right) dA = \iiint_{D} f(x,y) dA : \text{recover the previous formula}!$

* A bit more general: In APEX p.798, solid is above the graph $z = g(x,y)$ and below the graph $z = f(x,y)$ for $(x,y)$ in the domain $D$, then you get $\text{Vol} = \iiint_{D} (f(x,y) - g(x,y)) dA$. Just play the same game here:

solid: $g(x,y) \leq z \leq f(x,y)$ domain $D$ in $(x,y)$

$\iiint_{\text{solid}} 1 dV = \iiint_{D} \left( \int_{g(x,y)}^{f(x,y)} 1 \, dz \right) dA = \iiint_{D} (f(x,y) - g(x,y)) dA: \text{recover above formula}$
Extra problems left as homework:

A) Find volume of solid E bounded by the planes 
   \[ z = 6, \ y = 0, \ y - x = 3, \ \text{and} \ x + 2y + z = 3 \]

B) Final exam: problem 9 in 2014 - WT 1

C) Final exam: problem 7 in 2015 - WT 1 (in one of 
   the orders of integration, you'll actually need to break 
   into 2 integrals. There were < 5 students who could 
   (in all sections) do this correctly)

And after you finish cylindrical coordinates & spherical coordinates:

D) Final exam: problem 8 in 2015 - WT 1

E) Final exam: problem 7 in 2013 - WT 1

Do these yourselves first. I'll write the solution & explanation 
in a separate extra note.
Final remarks
- So, for the inner most \( \int_0^0 \) functions in other 2 variables
- the middle \( \int_0^0 \) functions in the remaining variable
- the outer most \( \int_0^0 \) constants

- I usually find a combination of rough picture + inequalities the most effective. All examples above (except the extra problems left as homework) are taken from the textbook, read them for other ways to explain the solution. (Stewart)

Stewart §15.8 SSS in Cylindrical Coordinates (or see Secondary Text #2)

* Cylindrical coordinates (centered at the z-axis)

Roughly: keep , describe ( ) in polar coordinates

Formal def: 
\( P(x,y,z) \) has cylindrical coordinate 
\( (r, \Theta, z) \) with 
\[ x = r \cos \Theta, \quad y = r \sin \Theta, \quad z = z \]
\( (0 \leq r \text{ and } 0 \leq \Theta \leq 2\pi \text{ as in §15.4, Stewart}) \)

Useful formulas: 
\[ x^2 + y^2 = r^2, \quad \tan \Theta = \frac{y}{x} \]
Eq

a) Plot the point with cylindrical coord \((e, \frac{2\pi}{3}, 1)\) in the \(xyz\)-coordinate system, then find the rectangular coordinate.

b) Find cylindrical coord. of \((1, -3, -7)\).

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Some surfaces in cylindrical coord.

Eq: surface \(r = 5\) in coord. \((r, \theta, z)\)?

What is it in \(xyz\)-coord?
Eq: surface \( r^2 + z^2 = 5 \) in coord \((r, \theta, z)\)?

What is it in \(xyz\)-coord?

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**Triple Integral using cylindrical coord:**

Suppose \( E \) has the type: \( u_1(x,y) \leq z \leq u_2(x,y) \)

\((x,y)\) in domain \(D\)

Use polar coord. to represent \(D\), say

\[ \alpha \leq \theta \leq \beta, \quad h_1(\theta) \leq r \leq h_2(\theta) \]

as in Stewart §15.4 on polar coord.
Then
\[ \iiint_E f(x, y, z) \, dV = \iint_D \left( \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right) \, dA \]
\[ = \rho \, d\theta \, dz \, dr \]
don't forget!

\text{Solid:} \quad \text{inside cylinder } x^2 + y^2 = 1
\text{below plane } z = 4 \quad \text{above the \ paraboloid } Z = 1 - x^2 - y^2

\text{Density at any point is proportional to its distance to the axis of the cylinder. Find mass (E)?}

\rightarrow \text{This is actually eq.3 in the book, show figure p.1054 Stewart (Stewart)}

\text{Solution.}