* Given $f(x,y)$: functions in 2 variables $x$ & $y$

**Informal rule & notation**

- The partial derivative of $f$ with respect to $x$:
  
  **Notation:** \( \frac{\partial f}{\partial x} \) or $f_x$

  **Rule:** regard $y$ as constant, differentiate with respect to $x$

- The partial derivative of $f$ with respect to $y$:
  
  **Notation:** \( \frac{\partial f}{\partial y} \) or $f_y$

  **Rule:** regard $x$ as constant, differentiate with respect to $y$

**Eq.1:** $f(x,y)$ has a nice formula:

\[ f(x,y) = xe^{xy} \]

a) In the "warm-up", we've found \( \frac{\partial f}{\partial y} \) (also written \( \frac{\partial}{\partial y}(xe^{xy}) \)) and get $xe^{xy}$. Now find \( \frac{\partial f}{\partial x} \) (also written \( \frac{\partial}{\partial x}(xe^{xy}) \))

b) Find $f_x(2,0)$ and $f_y(2,0)$.

**Solution:**
a) \( \frac{\partial f}{\partial y} = x^2 e^{xy} \). Now we find \( \frac{\partial f}{\partial x} \):
\[
\frac{\partial f}{\partial x} = \frac{3}{2} (x e^{xy}) \quad \text{product rule}
\quad e^{xy} + x \frac{\partial}{\partial x} (e^{xy})
\quad = e^{xy} + x e^{xy} \cdot y \quad \text{(there's a chain rule here)}
\quad = (1 + xy) e^{xy}
\]

b) Plug-in \( (x, y) = (2, 0) \):
\[
f_x (2, 0) = (1 + 0) e^0 = 1
\]
\[
f_y (2, 0) = 4 e^0 = 4
\]

\(\text{Eq.2: without a nice formula for } f, \text{ given } (a, b) \text{ (such as } (2, 0) \text{ in previous eq)}, \text{ how to find } \frac{\partial f}{\partial x} (a, b) \text{ } \frac{\partial f}{\partial y} (a, b) \) "theoretically"?

\(\text{Answer:}\)

For \( \frac{\partial f}{\partial x} (a, b) \): set \( g(x) = f(x, b) \)
which is a function in 1 variable \( x \)

Then \( \frac{\partial f}{\partial x} (a, b) = g'(a) \)
For \( \frac{df}{dy} (a, b) \): set \( l(y) = f(a, y) \) which is a function in 1 variable \( y \)

Then \( \frac{df}{dy} (a, b) = l'(b) \)

Eg in Stewart: \( f(T, H) \) with values in Table 1 p.925 Stewart

Approximate \( \frac{df}{dT} (g_4, 60) \)

answer: set \( g(T) = f(T, 60) \). Look at the column \( H = 60 \)
get the table of values for \( g(T) \):

\[
\begin{array}{cccc}
T & 90 & 92 & 94 & 96 \\
g(T) & 100 & 105 & 111 & 116
\end{array}
\]

answer \( \frac{df}{dT} (g_4, 60) = g'(g_4) \approx \frac{g(96) - g(94)}{96 - 94} = \frac{5}{2} \)

(also OK to take \( \approx \frac{g(92) - g(94)}{92 - 94} = \frac{6}{2} \); or even OK to take the average of \( \frac{5}{2} \times \frac{6}{2} \))
Formal definition:

\[ f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \]

\[ f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \]

Subscripts \( x, y \) & the \( x,y \) in bracket ( ) are different notation of partial der.

\( f_x, f_y \)

Other notations (besides \( \frac{\partial f}{\partial x}, f_x, \frac{\partial f}{\partial y}, f_y \)):

\( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \) with \( z = f(x,y) \)

\( D_z f, f_1, D_z f, f_2 \)

Interpretation of partial derivatives:

- As rate of change: \( z = f(x,y) \)
  \( \frac{\partial f}{\partial x}(a,b) \): rate of change of \( z \) in \( x \) at \( (a,b) \) when we fix \( y = b \) (or when moving along the trace/cross section in \( y = b \))
  \( \frac{\partial f}{\partial y}(a,b) \): similar

- As slopes of tangents of traces in \( x \) & traces in \( y \) (see p.928 Stewart)

Eg: Eg 406 p.695 APEX: explain this eg in the lecture

(Blue curve in Figure 12.12(a) is the trace/cross section in \( y = 1 \))
(Blue curve in Figure 12.12(b) is the trace/cross section in \( x = 2 \))
Geometric meaning of \( \frac{\partial f}{\partial x} (a, b) \) and \( \frac{\partial f}{\partial y} (a, b) \):

- Graph of \( f(x, y) \Rightarrow \text{surface } z = f(x, y) \)
- For \( \frac{\partial f}{\partial x} (a, b) \Rightarrow \text{fix } y = b \), look at the curve \( z = f(x, b) \)
  (-this curve is just the trace at \( y = b \))

\[
\text{graph } z = f(x, y) \\
\text{zoom} \\
\text{tangent}
\]

\[
\frac{\partial f}{\partial x} (a, b) = \frac{\partial z}{\partial x} (a, b) = \text{slope of the tangent of the curve } \text{where } x = a
\]

Similarly: for \( \frac{\partial f}{\partial y} (a, b) \Rightarrow \text{fix } x = a \Rightarrow \text{trace at } x = a \)
which is the curve \( z = f(a, y) \)

\[
\frac{\partial f}{\partial y} (a, b) = \frac{\partial z}{\partial y} (a, b) = \text{slope of the tangent when } y = b
\]
Eg: (compare eq 2 in p.928 of Stewart)

For the graph of \( f(x,y) = 4 - x^2 - y^2 \), find parametric equation for the tangent of the trace in \( y = 1 \) at the point \( P(1,1,2) \):

\[ P(1,1,2) \]

\[ (a,b) = (1,1) \]

\[ z = f(a,b) = f(1,1) = 2 \]

Answer:

The graph is the surface \( z = f(x,y) = 4 - x^2 - y^2 \) \( \left( \frac{df}{dx} = -2x \right) \)

Trace in \( y = 1 \) is the curve \( z = f(x,1) \)

Slope of the tangent line to this curve at \( P(1,1,2) \) is

\[ \frac{df}{dx} \bigg|_{(1,1)} = -2 \]

Parametric eq

\[ \begin{cases} x = 1 + t \\ y = 1 \\ z = 2 - 2t \end{cases} \]

(eq: \( \begin{cases} x - 1 = \frac{z - 2}{-2} \\ y = 1 \end{cases} \)

Why? Should have \( \begin{cases} x = 1 + at \\ y = 1 \\ z = 1 + ct \end{cases} \)

Your trace and tangent are in the plane \( y = 1 \)

Any \( a \times c \) with \( \frac{c}{a} = -2 \) is OK since \( \frac{c}{a} \) is the slope of the tangent of the trace \( z = f(x,1) \). Why \( \frac{c}{a} \) is slope?

from \( \frac{x - 1}{a} = \frac{z - 2}{c} \) \( \Rightarrow \) \( z = \left( \frac{c}{a} \right) x + \text{something} \)
Eq: Suppose \( x \) & \( y \) are variables and \( z \) is a function (implicitly) defined in \( x \) \& \( y \).

Find \( \frac{\partial}{\partial x} (xz + y\cos z) \)

**Solution:**

\[
\frac{\partial}{\partial x} (xz) + y \frac{\partial}{\partial x} (\cos z) = z + x \frac{\partial z}{\partial x} + y (-\sin z) \frac{\partial z}{\partial x}
\]

\[= z + (x - y\sin z) \frac{\partial z}{\partial x}\]

**Eq (page 929 Stewart)** \( z \) is defined implicitly in \( x \) \& \( y \) such that \( x^3 + y^3 + z^3 + 6xyz = 1 \)

Find \( \frac{\partial z}{\partial x} \) \& \( \frac{\partial z}{\partial y} \)

**Solution:** (this is exactly in p.929 Stewart). I'll find \( \frac{\partial z}{\partial x} \) here, it's similar for \( \frac{\partial z}{\partial y} \):

Start with \( x^3 + y^3 + z^3 + 6xyz = 1 \)

To make \( \frac{\partial z}{\partial x} \) appear, take \( \frac{\partial}{\partial x} \) both sides, get:

\[
3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6y (\frac{\partial}{\partial x} (xz)) = 0
\]

\[
3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6y (z + x \frac{\partial z}{\partial x}) = 0
\]

\[
3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0
\]

\[
(3z^2 + 6xy) \frac{\partial z}{\partial x} = -3x^2 - 6yz
\]

\[
\frac{\partial z}{\partial x} = \frac{-x^2 - 6yz}{3z^2 + 6xy}
\]
Functions of 3 or more variables:

- Similar def: regard all other vars as constant, differentiate with respect to one variable.

- Do not confuse with implicit differentiation:

Here \( f(x, y, z) \), \( \frac{\partial f}{\partial x} \), \( \frac{\partial f}{\partial y} \), \( \frac{\partial f}{\partial z} \): \( x, y, z \) have "equal status" as independent variables of \( f \),

previously in previous implicit diff: \( x \) & \( y \) are variables of \( z \)

so \( z \) depends on \( x \) and \( y \).

Eq: \( f(x, y, z) = xz + y\cos z \)

Find \( \frac{\partial}{\partial x} (xz + y\cos z) \), \( f_y \), \( \frac{\partial f}{\partial z} \) ?

Answer: (you should see why this is different from the previous page)

\[ \frac{\partial}{\partial x} (xz + y\cos z) = z + 0 = z \]

\[ f_y = \frac{\partial}{\partial y} (xz + y\cos z) = 0 + \cos z = \cos z \]

\[ \frac{\partial f}{\partial z} = \frac{\partial}{\partial z} (xz + y\cos z) = x - y\sin z \]