* Traces

- Why useful? Easier to visualize curves in $\mathbb{R}^2$ than visualizing surfaces in $\mathbb{R}^3$

- What are traces? (also called cross sections) (here: $k = \text{real } \neq 0$)

  **Trace in** $x = k$: curve obtained by intersecting/cutting the surface with the vertical plane $x = k$

  **Trace in** $y = k$: $y = k$

  **Trace in** $z = k$: horizontal plane $z = k$

**Eg:** show the 2 surfaces in p. 555 APEX

1) $z = y^2$

  **Trace in** $x = k$: parabola $\begin{cases} z = y^2 \\ x = k \end{cases}$

  **Trace in** $y = k$: one line $\begin{cases} y = k \\ z = k^2 \\ \text{x: anything} \end{cases}$ (this line is parallel to $\langle 1,0,0 \rangle$)

  **Trace in** $z = k$ (say $k > 0$): $\begin{cases} x: \text{anything} \\ y = \sqrt{k} \text{ or } -\sqrt{k} \\ z = k \end{cases}$ (2 lines parallel to $\langle 1,0,0 \rangle$)
2) $x^2 + y^2 = 1$

Trace in $x = k$
(say $-1 < k < 1$)

\[
\begin{align*}
&x = k \\
y = \sqrt{1-k^2} \text{ or } -\sqrt{1-k^2} \\
z : \text{ anything}
\end{align*}
\]

(2 lines parallel to $<0,0,1>$)

Trace in $y = k$
(say $-1 < k < 1$)

\[
\begin{align*}
&x = \sqrt{1-k^2} \text{ or } -\sqrt{1-k^2} \\
y = k \\
z : \text{ anything}
\end{align*}
\]

(2 lines parallel to $<0,0,1>$)

Trace in $z = k$

\[
\begin{align*}
x^2 + y^2 &= 1 \\
z &= k
\end{align*}
\]

Cylinders: above surfaces are examples of cylinders

Roughly: cylinder = take the curve defined by an eq. of 2 variables then move across the axis of the remaining variable

Eq: $z = y^2$ (move across $x$-axis)

$x^2 + y^2 = 1$ (move across $z$-axis)
Conic sections (0.1) & Quadric Surfaces

Conic sections
Roughly: curves in $\mathbb{R}^2$ given by polynomial equations of degree 2

3 types: 

- **Parabolas:** $y = x^2$ \quad $x = \frac{1}{2} y^2$ \quad $y = (x-1)^2 + 2$

- **Ellipses:** general eq: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

- **Hyperbolas:** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ \quad or \quad $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

Note: circles are special form of ellipses.
Note: sometimes, shifting is needed to have the above form

\[ \frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1 \quad \Rightarrow \quad \frac{X^2}{4} + \frac{Y^2}{3} = 1 \]

\[ \frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1 \quad \Rightarrow \quad \frac{Y^2}{9} - \frac{X^2}{4} = 1 \]

Quadric surfaces

Roughly: surfaces in \( \mathbb{R}^3 \) given by polynomial equations of degree 2

Classification:

1) Ellipsoid: all traces are ellipses

\[ \text{eq:} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \]

\[ \text{eq: show the ellipsoid} \quad x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1 \quad \text{in p.562 APEX} \]

2) Paraboloid: 2 types of traces are parabolas
2a) Elliptic paraboloid: the remaining traces are ellipses

\[ Z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

\[ \Rightarrow \text{show p. 559 APEX} \]

2b) Hyperbolic paraboloid: the remaining traces are hyperbolas

\[ Z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \]

\[ \Rightarrow \text{show p. 561 APEX} \]

3) Cones, Hyperboloids of One Sheet, Hyperboloids of Two Sheets

2 types of traces are hyperbolas

\[ \Rightarrow \text{show p. 559, 560, 561 APEX} \]

Eq: Show how to roughly visualize the surface

\[ y = \frac{x^2}{4} + \frac{z^2}{16} \quad (\text{this is an elliptic paraboloid}) \]

then show picture in p. 562
More pictures: see p. 584 in Stewart, do exercises 21-28 in Sec 12.6 of Stewart.

- Skip surface of revolution (you can read it yourselves)

## §12.1 Multivariable Functions

#### 2 variables:

- Domain $D$: subset of $\mathbb{R}^2$
- Every $(x, y)$ gives a value $f(x, y)$
  
  ![Graph](image)

**Graph of $f(x, y)$**: all points $(x, y, z)$ such that $z = f(x, y)$ and $(x, y)$ in $D$

**Draw picture**:
- The graph is the surface $z = f(x, y)$
- Graph $z = f(x, y)$
\[ f(x, y) = \sqrt{9 - x^2 - y^2} \]

a) Find the domain \( D \)

b) \( f(1, 2) = \) ?

c) What is the graph?

Answer:

a) Need \( 9 - x^2 - y^2 \geq 0 \), so \( x^2 + y^2 \leq 9 \)
   \( \rightarrow \) the domain \( D \) in \( \mathbb{R}^2 \) is the solid disk of radius 3 center \((0,0)\)

b) \( f(1, 2) = \sqrt{9 - 1 - 4} = 2 \)

c) Graph: points \((x, y, z)\) such that \( z = f(x, y) = \sqrt{9 - x^2 - y^2} \)
   \( \rightarrow \) \( x^2 + y^2 + z^2 = 9 \) and \( z \geq 0 \)
   \( \rightarrow \) upper hemisphere of radius 3 center \((0,0,0)\)
Previous eq: \( f(x,y) = \sqrt{9-x^2-y^2} \)

Require that the domain must satisfy

\[ x \geq 0, \ y \geq 0 \]

a) What is the domain?

b) Graph of \( f(x,y) \) under this more restricted domain?

(mention the word “first octant” here!)

Solution:

a) Solid disk \( x^2 + y^2 \leq 9 \) in the first quadrant \( x \geq 0, \ y \geq 0 \)

\[
\begin{align*}
\text{D}
\end{align*}
\]

b) \( \frac{1}{8} \) of the sphere of radius 3 centered at \((0,0,0)\) in the “first octant”

(THIS MEANS THE PART OF \( \mathbb{R}^3 \) WITH \( x \geq 0, \ y \geq 0, \ z \geq 0 \))