§10.6 Planes

How to determine a plane in $\mathbb{R}^3$?

* Obvious answer: given 3 non-collinear points
* More convenient answer: given a point on the plane and a normal vector of the plane

(here: normal vector of a plane is a vector that is orthogonal to the plane)

Problem: given $P(x_0, y_0, z_0)$ and $\vec{n} = \langle a, b, c \rangle$.

Describe the plane $S$ containing $P$ and orthogonal to $\vec{n}$.

Answer: describe random point $Q(x, y, z)$ on $S$.

Relation $\overrightarrow{PQ}$ and $\vec{n}$?

$\overrightarrow{PQ} \perp \vec{n}$ or $\overrightarrow{PQ} \cdot \vec{n} = 0$

Vector eq: (we $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$)

$(\overrightarrow{OQ} - \overrightarrow{OP}) \cdot \vec{n} = 0$ or $\langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle a, b, c \rangle = 0$
Equations of a plane:

- Scalar equation (also called "standard form" in the book):
  \[ a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \]

- Linear equation:
  \[ ax + by + cz + d = 0 \]
  where \( d = -ax_0 - by_0 - cz_0 \)

(note: slightly differs from the textbook: \( ax + by + cz = d \)
  where \( d = ax_0 + by_0 + cz_0 \) )

Find eq of plane through 3 points

Eq: find eq for the plane through \( P(1,3,2), Q(3,-1,6), R(5,2,0) \)

(Hint: need \( \vec{n} \). Know: \( \vec{n} \perp \) any vector formed by \( P, Q, R \))

Answer:
\[
\begin{align*}
\vec{PQ} &= \langle 2, -4, 4 \rangle \\
\vec{PR} &= \langle 4, -1, -2 \rangle
\end{align*}
\]

then \( \vec{n} = \vec{PQ} \times \vec{PR} = \left| \begin{array}{ccc} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{array} \right| = \langle 12, -20, 14 \rangle \)

Answer: \( 12(x-1) + 20(y-3) + 14(z-2) = 0 \)

Find eq for the line that is the intersection of 2 planes
Method: pick a point \( P \) in the intersection, then

Method 1: find \( \overrightarrow{d} \) parallel to the intersection line
Know: \( \overrightarrow{d} \perp \overrightarrow{n_1} \) and \( \overrightarrow{d} \perp \overrightarrow{n_2} \) \( \Rightarrow \) choose \( \overrightarrow{d} = \overrightarrow{n_1} \times \overrightarrow{n_2} \)

Method 2: pick another point \( Q \) \( \Rightarrow \) line through \( P \) & \( Q \)

Eq: Find symmetric eq for the intersection of the planes

\( S_1: \ x + y + z = 1 \)

and \( S_2: \ x - 2y + 3z = 1 \)

answer: pick \( P \) in the intersection; solve \( \begin{cases} x + y + z = 1 \\ x - 2y + 3z = 1 \end{cases} \)

We only need one point \( P \), set \( x = 1 \) & solve for \( (y, z) \) (you can set \( x = 100 \) & solve for \( (y, z) \) or set \( y = 0 \) & solve for \( (x, z) \), etc.)

\( \begin{cases} 1 + y + z = 1 \\ 1 - 2y + 3z = 1 \end{cases} \)

\( \Rightarrow y = z = 0 \)

\( P(1, 0, 0) \)

Method 1: \( \overrightarrow{n_1} = \langle 1, 1, 1 \rangle \), \( \overrightarrow{n_2} = \langle 1, -2, 3 \rangle \)

\( \overrightarrow{d} = \overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = \langle 5, -2, -3 \rangle \)

answer: \( \frac{x - 1}{5} = \frac{y}{-2} = \frac{z}{-3} \)

Method 2: pick another \( Q \): set \( x = 0 \) & solve for \( (y, z) \):

\( \begin{cases} y + z = 1 \\ -2y + 3z = 1 \end{cases} \)

\( \frac{-2(1-z) + 3z = 1}{z = \frac{3}{5} \quad , \quad y = \frac{2}{5}} \)

\( Q(0, \frac{3}{5}, \frac{2}{5}) \)
take \( \vec{d} = \overrightarrow{PA} = \langle -1, \frac{2}{5}, \frac{3}{5} \rangle \)

answer: \( \frac{x-1}{-1} = \frac{y}{\frac{2}{5}} = \frac{z}{\frac{3}{5}} \)
(you should see that this answer and the one from method 1 are the same)

\* Distance from a point to a plane (or distance between 2 parallel planes)

Find distance from C to the plane S

\[ \text{Pick any } P \text{ on } S \]

Here distance = length of DC

\[ \overrightarrow{DC} = \text{proj}_n \overrightarrow{PC} \]

\[ \Rightarrow \text{distance} = \text{length of } \text{proj}_n \overrightarrow{PC} = \frac{|\overrightarrow{PC} \cdot \vec{n}|}{||\vec{n}||} \]
(this is Key Idea 5.1 in p.620)

A useful formula (that the book doesn't give you):

If S is given by the eq: \( ax + by + cz + d = 0 \) and \( C(x_1, y_1, z_1) \) then the distance from C to S is

\[ \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} \]
Eq: Given 2 planes:

\[ S_1: \quad 10x + 2y - 2z = 5 \]
\[ S_2: \quad 5x + y - z = 1 \]

a) Why are they parallel?
b) Distance between them?

Answer: a) \( S_1 \) is parallel to \( S_2 \) since the normal vector \( \vec{n}_1 = \langle 10, 2, -2 \rangle \) is parallel to \( \vec{n}_2 = \langle 5, 1, -1 \rangle \).

b) Pick any point \( C \) on \( S_1 \), find distance from \( C \) to \( S_2 \).
To pick \( C \), need \( 10x + 2y - 2z = 5 \).
Say: choose \( y = z = 0 \Rightarrow x = \frac{1}{2} \) \( \Rightarrow C \left( \frac{1}{2}, 0, 0 \right) \)
\[ S_2: \quad 5x + y - z - 1 = 0 \] use the given formula for distance from \( C \) to \( S_2 \):
\[
\frac{|\frac{5}{2} + 0 - 0 - 1|}{\sqrt{5^2 + 1^2 + (-1)^2}} = \frac{3}{2 \sqrt{27}}
\]

* Angles: (not in textbook, could be asked in Webwork, exams...)
Rule: use acute angles (i.e. \( \leq 90^\circ \))
- Draw picture, use normal vectors, and see.

* Angle between 2 planes:
In these pictures, the angle between 2 planes should be $\theta_1$.

It turns out $\theta_1 = \theta_2$.

And $\theta_2$ is “cut out” by $\vec{n}_1$ and $\vec{n}_2$.

$\Rightarrow \quad$ the angle between 2 planes is the **acute angle** formed by 2 normal vectors

Eq: Find the angle between the planes $x+2y+3z = 0$ and $3x-4y = 1$

$\vec{n}_1 = \langle 1, 2, 3 \rangle$, $\vec{n}_2 = \langle 3, -4, 0 \rangle$

$\Theta = \text{angle between } \vec{n}_1 \text{ and } \vec{n}_2$, $\cos \Theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{\|n_1\| \|n_2\|} = \frac{-5}{\sqrt{14} \sqrt{25}} = -\frac{1}{\sqrt{14}}$

So $\Theta = \arccos \left(-\frac{1}{\sqrt{14}}\right)$

Warning: since we want an acute angle, the final answer is $\pi - \arccos \left(-\frac{1}{\sqrt{14}}\right)$ (which is also $\arccos \left(\frac{1}{\sqrt{14}}\right)$)
Angle between a line & a plane

Find $\Theta_2$ = acute angle between $L$ and $\vec{n}$

answer: angle between $L$ and $S$ is $\Theta_1 = \frac{\pi}{2} - \Theta_2$

Eq: find the angle between the plane $3x - 4y = 1$ and the line $x - 1 = \frac{y}{2} = z$

answer: first, find the acute angle cut out by $L$ and $\vec{n} = \langle 3, -4, 0 \rangle$. A vector parallel $L$ is $\vec{d} = \langle 1, 2, 1 \rangle$

$\Theta = \text{angle between } \vec{n} \text{ and } \vec{d}$

$\cos \Theta = \frac{\vec{n} \cdot \vec{d}}{||\vec{n}|| ||\vec{d}||} = \frac{-5}{\sqrt{25} \sqrt{6}} = \frac{-1}{\sqrt{6}}$

the acute angle is $\arccos \left( \frac{1}{\sqrt{6}} \right)$

answer: $\frac{\pi}{2} - \arccos \left( \frac{1}{\sqrt{6}} \right)$
Cylinders and Quadric Surfaces

(Skip surfaces of revolution; Sec 10.4 in Apex book)
Sec 12.6 in Stewart

Warm-up: In $\mathbb{R}^3$

a) Does the eq. $x^2 + y^2 = 1$ represent a circle?

b) What is it and sketch it?

Answer: a) No. In $\mathbb{R}^2$, $x^2 + y^2 = 1$ represents the unit circle $C$; but we are now in $\mathbb{R}^3$, the coordinate $z$ can be anything.

b) It’s the surface obtained when we “move” the circle $C$ across the $z$-axis (since $z$ can be anything)

$\Rightarrow$ an infinite cylinder

Our goal:

- Learn how to sketch cylinders
- Sketch surfaces using traces in $x, y, z$
- Recall conic sections: ellipses, parabolas, hyperbolas
- Sketch & visualize ellipsoids and simple elliptic paraboloids such as $y = 2x^2 + z^2$, etc.