Eg 8: sketch a plane in $\mathbb{R}^3$

a) Need how many non-collinear points to determine a plane?

\textit{Ans:} 3 points (then get a triangle & think of the plane spread out by this triangle)

b) Sketch the plane rep. by $x + y + z = 1$

\textit{Ans:} pick 3 points, say $(1,0,0), (0,1,0)$, and $(0,0,1)$

\[ \begin{array}{c}
\text{the origin 0} \\
\text{is actually behind this plane}
\end{array} \]

Eg 9 (a bit harder than previous ones)

a) Given 2 points $A$ and $B$ in $\mathbb{R}^3$, can you guess the set of points $P$ which are equidistant to $A$ and $B$?

b) let $A(1,-2,3), B(0,1,1)$, write down an equation representing the set of points $P(x,y,z)$ that are equidistant to $A$ and $B$.

(Hint: set $\|PA\| = \|PB\|$, square both sides, then simplify)

\textit{Answer:} a) think a bit and we guess that it
a) Think a bit and we guess that it should be a plane (that lies exactly in the middle and perpendicular to AB)

b) Set \( \|PA\| = \|PB\| \)

\[
\|PA\|^2 = \|PB\|^2
\]

\[
(x-1)^2 + (y+2)^2 + (z-3)^2 = x^2 + (y-1)^2 + (z-1)^2
\]

\[
x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 - 6z + 9
\]

\[
= x^2 + y^2 - 2y + 1 + z^2 - 2z + 1
\]

Simplify:

\[
-2x + 6y - 4z + 12 = 0
\]
§ 10.2 Vectors

* Basic definition.

- Vector: a line segment with a direction
- P: initial point
- Q: terminal point
- Length of the vector = length of the segment $\|\overrightarrow{PQ}\|$.

- Two vectors are equal if they have the same direction and same length (need not coincide).

Example: $\overrightarrow{u} = \overrightarrow{v}$

Add two vectors:

\[
\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}
\]
Q: How about $\vec{u} + \vec{v}$ where

```
\[ \vec{u} \]
\[ \vec{v} \]
```

Answer: "move $\vec{v}$"

1. \[ \vec{u} \rightarrow \vec{u} + \vec{v} \]

or

2. \[ \text{Parallelogram law:} \]

```
\[ \vec{u} \]
\[ \vec{v} \]
\[ \vec{u} + \vec{v} \]
```

Multiply a (scalar) real #

$\vec{c}$: real #, $\vec{v}$: vector

$\vec{c}\vec{v}$

- same direction as $\vec{v}$ if $c > 0$
- opposite direction to $\vec{v}$ if $c < 0$
- the zero vector if $c = 0$

\[
\text{length of } \vec{c}\vec{v} = |c| \cdot \text{length of } \vec{v}
\]

In our notation:

\[
|\vec{c}\vec{v}| = |c| |\vec{v}|
\]

* Vector addition & scalar multi.:

Using coordinates for easy calculation (very important) called "components" in the book

In $\mathbb{R}^2$: given $\vec{v}$, move $\vec{v}$ so that origin = initial, call terminal = $P$

$\Rightarrow$ the coordinates of $P$ represent $\vec{v}$
Eg.1: \( \vec{v} \) with initial \((1,2)\) terminal \((4,3)\)

What is the coordinate representation (also called component form) of \( \vec{v} \)?

Answer: \( <4-1, 3-2> = <3, 1> \)

Notation \( <3, 1> \) does not mean the point \( P(3,1) \) (note the pointed bracket)

It means the vector \( \overrightarrow{OP} \)

Formula: \( A(x_1, y_1), \ B(x_2, y_2) \) then \( \overrightarrow{AB} \) has component form

\[
\overrightarrow{AB} = <x_2-x_1, y_2-y_1>\

Addition, scalar mult., & length using component form:
\[ \vec{a} = \langle a_1, a_2 \rangle, \quad \vec{b} = \langle b_1, b_2 \rangle, \quad c: \text{ real} \]
\[ \vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \]
\[ c\vec{a} = \langle ca_1, ca_2 \rangle \]
\[ \text{length of } \vec{a} \quad ||\vec{a}|| = \sqrt{a_1^2 + a_2^2} \]

* Similar discussion & formulae for vectors in \( \mathbb{R}^3 \)

Eq: Determine if points lie on straight line (without drawing)
a) \( A(2,4,2), \quad B(3,7,-2), \quad C(1,3,3) \)

Answer: (idea: check if \( \vec{AB} \) and \( \vec{AC} \) on the same line (“parallel” to each other))
- check if \( \vec{AB} = c \vec{AC} \) for some scalar \( c \)
- check if coordinates of \( \vec{AB} \) & \( \vec{AC} \) are proportional, comment quotient = \( c \)

\[ \vec{AB} = \langle 1, 3, -4 \rangle \]
\[ \vec{AC} = \langle -1, -1, 1 \rangle \]
Is there \( c = \frac{3}{-1} \times \frac{-4}{4} = -1 \)
No, these 2 quotients are different
\( \Rightarrow \vec{AB} \) and \( \vec{AC} \) are not parallel \( \Rightarrow \) \( A, B, C \) not on straight line.
b) \(D(0,-5,5), \ E(1,-2,4), \) and \(F(3,4,2)\)

\[
\overrightarrow{DE} = \langle 1,3,-1 \rangle, \ \overrightarrow{DF} = \langle 3,9,-3 \rangle
\]

common quotient \(c = \frac{1}{3} = \frac{3}{9} = -\frac{1}{3}\), so \(\overrightarrow{DE} = \frac{1}{3} \overrightarrow{DF}\)

\(\Rightarrow\) \(\overrightarrow{DE}\) and \(\overrightarrow{DF}\) are parallel (we even know \(E\) lies in between \(D\) & \(F\))

\(\Rightarrow\) \(D, E, F\) on same line (since \(D \rightarrow E \rightarrow F\))

\[\text{Eq: Find the midpoint of } AB \text{ in the previous eg}\]

\[
\text{Hint: } \overrightarrow{OM} = \begin{bmatrix} \frac{1}{2} \end{bmatrix}, \ \text{the diagonal } \overrightarrow{OG} = \begin{bmatrix} \frac{1}{2} \end{bmatrix} \cdot \left( \overrightarrow{OA} + \overrightarrow{OB} \right)
\]

\[\Rightarrow \text{coordinate of } M = \left( \frac{3}{2} \right) \left( \text{coord.of } A + \text{ coord.of } B \right)\]

\[\text{Answer: } M = \left( \frac{2+3}{2}, \frac{4+7}{2}, \frac{2+(-2)}{2} \right) = \left( \frac{5}{2}, \frac{11}{2}, 0 \right)\]

\[\text{Properties of Vector Operations: see Theorem 84 in p.572 of the primary textbook}\]

\[\text{Roughly: all the natural properties that you expect}\]

\[\text{Unit vectors}\]

\[\text{Def: unit vector is a vector of length 1}\]
* given \( \overrightarrow{v} = \overrightarrow{0} \), the zero vector \( (0,0) \) in \( \mathbb{R}^2 \), \( (0,0,0) \) in \( \mathbb{R}^3 \) 

the unit vector pointing

. in the same direction of \( \overrightarrow{v} \) is \( \frac{\overrightarrow{v}}{||\overrightarrow{v}||} \) (divide the length to get length 1)

. in the opposite direction of \( \overrightarrow{v} \) is \( -\frac{\overrightarrow{v}}{||\overrightarrow{v}||} \)

eg: find the unit vector in the direction of \( \langle 1, 2, -3 \rangle \)

ans: length = \( \sqrt{1 + 4 + 9} = \sqrt{14} \)

answer: \( \langle \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \rangle \)

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**Standard Unit Vectors**

In \( \mathbb{R}^3 \), let

\( \overrightarrow{i} \) means \( \langle 1, 0, 0 \rangle \) (there are unit vectors in the positive \( Ox, Oy, Oz \) direction)

\( \overrightarrow{j} \) means \( \langle 0, 1, 0 \rangle \)

\( \overrightarrow{k} \) means \( \langle 0, 0, 1 \rangle \)

eg: \( 10 \overrightarrow{k} + 100 \overrightarrow{j} - 1000 \overrightarrow{i} \)

\[ = \langle -1000, 100, 10 \rangle \]

Conversely:

\( \langle 3, 4, 5 \rangle = 3 \overrightarrow{i} + 4 \overrightarrow{j} + 5 \overrightarrow{k} \)

\( \langle a_1, a_2, a_3 \rangle = a_1 \overrightarrow{i} + a_2 \overrightarrow{j} + a_3 \overrightarrow{k} \)

\( \langle 0, 1, 0 \rangle + \langle 0, 100, 0 \rangle + \langle -1000, 0, 0 \rangle \)
Eq: (a bit tricky, do it yourself, not discussed in the lecture)

In $\mathbb{R}^2$, let $A(-1,3)$ and $B(5,2)$. Find a point $C$ such that the line through $O$ and $C$ bisect the angle $\angle AOB$

Idea: it is easy to calculate the midpoint $M$, BUT the line through $O \pm M$ does not bisect the angle UNLESS the triangle were isosceles (UNLESS $\|OA\| = \|OB\|$)

$\Rightarrow$ keep the angle, "make $\|OA\| = \|OB\|"$, then take midpoint.

Solution: replace $\overrightarrow{OA}$ by the corresponding unit vector

$\overrightarrow{OA}_1 = \frac{\overrightarrow{OA}}{\|\overrightarrow{OA}\|} = \left< \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right>$

Similarly, replace $\overrightarrow{OB}$ by $\overrightarrow{OB}_1 = \left< \frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}} \right>$

Now $\|\overrightarrow{OA}_1\| = \|\overrightarrow{OB}_1\| = 1$

take $C$ = midpoint of $A_1B_1$

$C = \left( \frac{1}{2} \left( \frac{-1}{\sqrt{10}} + \frac{5}{\sqrt{29}} \right), \frac{1}{2} \left( \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{29}} \right) \right)$

$\Rightarrow$ you can also solve this using dot product