Eg: Previous \( f(x,y) = x^4 + y^4 - 4xy + 1 \). Try to find / guess global max & global min?

Answer: *No global max since there's no local max.

(Assume global min exists) we have that global min is \( \min(\text{all local min}) = -1 \) obtained at \((1,1)\) \& \((-1,-1)\)

Note: unless we explicitly ask you to do so, do not attempt to explain why global min/max exists or does not exist.

Eg: (Exercise 42 Page 979) Find points on the surface \( y^2 = 9 + xz \) that are closest to the origin & find this smallest distance.

Solution: need to find \( \min \) of \( \sqrt{x^2 + y^2 + z^2} \) under "constraint" \( y^2 = 9 + xz \).

2 methods:

- Method 1: solve for 1 variable \( \Rightarrow \) plug-in get a function in 2 vars without constraint

- Method 2: Lagrange multiplier in §14.8 Stewart (secondary textbook)

Let's do method 1:

Find \( \min \) of \( f(x,y,z) = x^2 + y^2 + z^2 \) (slightly nicer than \( \sqrt{x^2 + y^2 + z^2} \); at the end, take \( \sqrt{...} \) of your answer)

Plug-in \( y^2 = 9 + xz \), we will minimize \( h(x,z) = x^2 + 9 + xz + z^2 \)

Find critical pt: solve \( \begin{cases} h_x = 2x + z = 0 \\ h_z = x + 2z = 0 \end{cases} \)

\( \begin{cases} z = -2x \\ x - 4x = 0 \end{cases} \) \( x = 0 \) \& \( z = 0 \)

(without any constraint)

Well, technically there's a mild one: \( 9 + xz \) can't be \(< 0 \) so that we can find y
only 1 critical point \((0,0)\)

you can use the 2nd deriv test and see that \((0,0)\) gives you a (local) min of \(g(x,z)\). However since this is the only critical point and we know that there's an answer to the original question (i.e. global min exists) you can simply conclude the solution, as follows.

\[
g(0,0) = 9
\]

Answer: \(x = 0, z = 0 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3\)

\[
\Rightarrow \text{ two points } (0,3,0) \text{ and } (0,-3,0), \text{ and the shortest distance is } \sqrt{9} = 3.
\]

Absolute min/max over a closed & bounded set

Recall: \(f(x)\) 1-var, find absolute min/max on \([a,b]\)

Closed Interval Method: \(\star\) find critical points on \((a,b)\)

\(\star\) find values of \(f\) at those critical pts and at \(a \& b\)

\[
\Rightarrow \min = \min \text{ (those values)}, \max = \max \text{ (those values)}
\]

Back to \(\mathbb{R}^2\): analogue of \([a,b]\) will be "closed & bounded subset", analogue of \(a \& b\) will be "boundary".

Ad hoc: Not closed: if you can walk to a point not in \(D\) using points in \(D\).
eq: "solid" circle minus one point; "solid" circle minus a segment

\[ \text{Closed or } \text{Not closed} \]

\[ \text{Closed or } \text{Not Closed ?} \]

⇒ explain: walk to this excluded point using points in D

eg of closed sets:

"solid" circle
circle

rectangle / triangle

are closed

Bounded: D is bounded if we can put it inside some (large) disk

Quick question: which of the above closed sets are not bounded? Answer: line
Problem (2d version of Closed Interval Problem)
\[ f(x, y) \text{, domain } D \text{ is closed & bounded} \]
Find absolute max/min

Method:
1) Find critical values in \( D \)
2) Find absolute max/min on the boundary of \( D \)
3) \( \text{abs min} = \min (\text{above values}) \)
   \( \text{abs max} = \max (\text{above values}) \)

Extra explanation
- Boundary: points where we can walk to from both
  (think of \( D \): house
  boundary = outer wall, floor, roof)

- Ad hoc: how to recognize closed, bounded, boundary...
  → use picture
  → \( D \) is closed if it is given by no strict inequality
    (always "=" or "\( \leq \)" or "\( \geq \)"")
boundary: set the "≤" and "≥" to equality.

final remark: on boundary, get constraint on x & y, use either solve for 1-var, then plug-in or Lagrange multiplier in §14.8

abs max/min of \( f(x,y) = x^2 - 2xy + 2y \) on rectangle \( D = \{(x,y) | 0 ≤ x ≤ 3, 0 ≤ y ≤ 2\} \)

Solution: (expect a long solution)

Step 1: Find critical values

Solve \( \begin{cases} f_x = 2x - 2y = 0 \\ f_y = -2x + 2 = 0 \end{cases} \Rightarrow (x=1, y=1) \) (take this since it's in D)

\( f(1,1) = 1 \)

Step 2: Find max/min on boundary

On \( B_1: x = 0, 0 ≤ y ≤ 2 \) plug-in f, get the function \( 2y \).

So \( \max f(0,2) = 4 \) on \( B_1 \)

\( \min f(0,0) = 0 \) on \( B_1 \)

On \( B_2: y = 0, 0 ≤ x ≤ 3 \) plug-in f, get the function \( x^2 \).

So \( \max f(3,0) = 9 \) on \( B_2 \)

\( \min f(0,0) = 0 \) on \( B_2 \)
On $B_3: x = 3, 0 \leq y \leq 2$
Plug-in $f$, get the function $9 - 4y$
So $\max f(3,0) = 9$ on $B_3$
$\min f(2,0) = 1$ on $B_3$

On $B_4: y = 2, 0 \leq x \leq 3$
Plug-in $f$, get $x^2 - 4x + 4$. Since this is a bit more complicated than other previous functions ($2y, x^2, 9 - 4y$), I'll do closed interval method here:

$g(x) = x^2 - 4x + 4$ for $0 \leq x \leq 3$
$g'(x) = 2x - 4 = 0$ when $x = 2$

$\begin{align*}
    f(2,2) &= g(2) = 0 \\
    f(0,2) &= g(0) = 4 \\
    f(3,2) &= g(3) = 1
\end{align*}$

$\rightarrow \min$ on $B_4$
$\rightarrow \max$ on $B_4$

Overall: compare all the values in the boxes, conclusion:
$\max = 9$ obtained at $(3,0)$
$\min = 0$ obtained at $(0,0)$ & $(2,2)$.

Summary & final remarks: (read this yourself at home)

* We have discussed 2 types of problems ($\S$ 14.7 Stewart or $\S$ 12.8 APEX)
  (1) Find and classify critical points.
  (2) Find global max/min over a closed & bounded domain.

* For Type (1): need 2nd der. test to classify critical points.
  For Type (2): you know global max/min always exist (p.975 Stewart),
  just find the critical points in $D$, their values, max/min on boundary,
  then compare $\Rightarrow$ no need to use 2nd der. test.

* On boundary, you have "constraint", you can either
  - Solve for 1 variable, plug-in ... (as in a previous eg.)
  - This is not always possible, so there's another method: Lagrange multiplier ($\S$ 14.8 Stewart or Secondary textbook #1)

* Sometimes, you don't have one single formula to describe the whole boundary $\Rightarrow$ need to break the boundary
E.g.: previous rectangle e.g., break to 4 pieces
\[
\begin{cases}
x=0, & 0 \leq y \leq 2 \\
x=3, & 0 \leq y \leq 2 \\
y=0, & 0 \leq x \leq 3 \\
y=2, & 0 \leq x \leq 3
\end{cases}
\]

E.g.: if \( D \) is the (solid) disk \( x^2 + y^2 \leq 1 \), the boundary can be nicely expressed as: \( x^2 + y^2 = 1 \) (so we don't need to "break it" as in the above rectangle e.g.)

* This is not mentioned in the book but, except for 2nd der. test, everything holds similarly for functions in 3 or more variables:

E.g.: \( f(x,y,z) \), to find crit. pt, solve
\[
\begin{cases}
f_x = 0 \\
f_y = 0 \\
f_z = 0
\end{cases}
\]

E.g.: find \( \text{max/min of } f(x,y,z) \) in the (closed & bounded) domain \( x^2 + y^2 + z^2 \leq 1 \) (this is the solid sphere).
- Find critical points, then values of \( f \) at crit. points.
- Find max/min on the boundary \( x^2 + y^2 + z^2 = 1 \)
- Take max/min of all the above values.

Well, there's actually a 2nd der. test for 3 or more variables, but it's way more complicated and lies outside the scope of our class.