**Topic:** Max & Min of Multivariable Functions

**Overview:**

- We will solve 3 types of optimization problems:
  1. Find & classify critical points
  2. Find max/min over a "closed & bounded" domain
  3. Find max/min under a constraint using the method of Lagrange multipliers.

- 2 comments
  - Once you learn everything, you'll see that (2) is sort of a combination of (1) & (3)

- Textbooks:
  - Stewart treats all 3 problems (Sec 14.7 & 14.8)
  - APEX only treats (1) & (2), unfortunately
  - Refer to Secondary Textbook #1 (Community Calculus) for (3).
* Understand the difference

1) **global/absolute max:** the biggest value of the function over the whole domain

2) **global/absolute min:** the smallest value of the function over the whole domain

*local/relative max:** the biggest value of the function over some small neighborhood.

*local/relative min:** the smallest value of the function over some small neighborhood.

* Show picture in p.970 Stewart

* Critical points:

  - Recall previous calculus:
    - \( f(x) \): function in 1 variable. Local max/min only possible at **critical points** either \( f'(x) \) does not exist or \( f'(x) \) exists and = 0.
• Back to Math 200: \( f(x,y) \): function in 2 variables 
\((a,b)\) is a critical point if 
either at least one of \( f_x(a,b) \) & \( f_y(a,b) \): does not exist. 
or \( f_x(a,b) \) & \( f_y(a,b) \) exist and both = 0

In our class, \( f \) (almost always) has a nice formula and 
\( f_x \) & \( f_y \) exist \( \Rightarrow \) only care about the boxed condition. 

Repeat: (see Theorem 2 in p.970 Stewart) 
\[ f \text{ has local max/min at } (a,b) \text{ and } f_x(a,b), f_y(a,b) \text{ exist,} \]
then \( f_x(a,b) = f_y(a,b) = 0 \).

Some geometric intuition: graph \( z = f(x,y) \) at local max/min 
\((a,b)\):

• What does tangent plane look like? horizontal tangent plane (parallel xy-plane, equation: \( z - z_0 = 0 \)).
• Why \( \nabla f(a,b) = \overrightarrow{0} \), what if \( \nabla f(a,b) \neq \overrightarrow{0} \)?
Max \( D_u f(a,b) = \nabla \| \nabla f(a,b) \| > 0 \Rightarrow f \) increasing \( \Rightarrow \) can't be max.
Min \( D_u f(a,b) = -\| \nabla f(a,b) \| < 0 \Rightarrow f \) decreasing \( \Rightarrow \) can't be min.
Eq: \( f(x,y) = x^2 + y^3 - 2x - 6y + 14 \)

a) Find the points \((a,b)\) where \(f\) might have local max or min.

b) Soon you'll be able to explain why the point in part (a) gives you a \underline{local min}. Can you explain why it's even a \underline{global min} (this usually requires some effort)?

Solution

a) \( f_x = 2x - 2, \ f_y = 2y - 6 \)

Solve the system \( \begin{cases} f_x = 2x - 2 = 0 \\ f_y = 2y - 6 = 0 \end{cases} \) get \( \begin{cases} x = 1 \\ y = 3 \end{cases} \Rightarrow \ (1,3) \)

b) \( f(x,y) = x^2 - 2x + 1 + y^2 - 6y + 9 + 4 \)

\[ = (x - 1)^2 + (y - 3)^2 + 4 \]

\[ \geq 4 \]

\[ f(1,3) = 4 \]

So \underline{global min is 4 obtained at \((1,3)\).}

Recall: (previous calculus) \( f(x) \): 1-var. Once we know \( f'(a) = 0 \), how to determine if \( a \) is a local min/max?

\( 2^\text{nd} \) derivative test (1-variable)

If \( f''(a) > 0 \) : local min

\( f''(a) < 0 \) : local max

\( f''(a) = 0 \) : inconclusive (could be "anything")
Second Derivative Test (p. 974): \( f(x,y) \) : 2-vari

Know \( f_x(a,b) = f_y(a,b) = 0 \). let

\[
D = f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2
\]

If

- \( D > 0 \) and \( f_{xx}(a,b) > 0 \): \underline{local min} \( f(a,b) \) at \((a,b)\).
- \( D > 0 \) and \( f_{xx}(a,b) < 0 \): \((a,b)\) gives \underline{local max} \( f(a,b) \).
- \( D < 0 \): \underline{saddle point} (neither local max nor min)

(show picture of saddle point in p.974)

Remark: if \( D = 0 \): the test is inconclusive, \((a,b)\) could give you anything \(\leftarrow\) \underline{local max}
\underline{local min} \(\rightarrow\) \underline{saddle}

Typical/simlest eg. classify the critical points of

\[ a) \quad f(x,y) = x^2 + y^2 \]

Solve \( \begin{cases} f_x = 2x = 0 \Rightarrow \text{only crit. } (0,0) \\ f_y = 2y = 0 \end{cases} \)

\( f_{xx} = 2, \ f_{yy} = 2, \ f_{xy} = 0 \)

\( \Rightarrow D = 4 > 0, \ f_{xx} = 2 > 0 \): \underline{local min}

\[ c) \quad f(x,y) = x^2 - y^2 \]

Solve \( \begin{cases} f_x = 2x = 0 \Rightarrow \text{only crit. } (0,0) \\ f_y = -2y = 0 \end{cases} \)

\( f_{xx} = 2, \ f_{yy} = -2, \ f_{xy} = 0 \)

\( D = -4 < 0 \Rightarrow \underline{saddle point} \)

\( b) \quad f(x,y) = -x^2 - y^2 \)

Solve \( \begin{cases} f_x = -2x = 0 \Rightarrow \text{only crit. } (0,0) \\ f_y = -2y = 0 \end{cases} \)

\( f_{xx} = -2, \ f_{yy} = -2, \ f_{xy} = 0 \)

\( \Rightarrow D = 4 > 0, \ f_{xx} = -2 < 0 \): \underline{local max}

(note: once \( D < 0 \), you \underline{don't care about} \( f_{xx} > 0 \) or \(< 0 \), always saddle point)
Example: find & classify the critical points of
\[ f(x,y) = x^4 + y^4 - 4xy + 1 \]

Solution: solve the system:
\[
\begin{align*}
\frac{\partial f}{\partial x} &= 4x^3 - 4y = 0 \\
\frac{\partial f}{\partial y} &= 4y^3 - 4x = 0
\end{align*}
\]
\[ \begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases} \]
\[ \begin{cases} x(x^8 - 1) = 0 \Rightarrow x = 0, \pm 1 \\ y = x^3 \end{cases} \]
Get 3 critical points: \((0,0), (1,1), (-1,-1)\)

\[ f_{xx} = 12x^2, \quad f_{yy} = 12y^2, \quad f_{xy} = -4 \]
\[ D(x,y) = 144x^2y^2 - 16 \]

At \((0,0)\): \( D(0,0) = -16 < 0 \Rightarrow (0,0) \) gives a saddle point.

At \((1,1)\): \( D(1,1) = 144 - 16 > 0 \), \( f_{xx}(1,1) = 12 > 0 \)
\( \Rightarrow (1,1) \) gives the local min \( f(1,1) = -1 \)

At \((-1,-1)\): \( D(-1,-1) = 144 - 16 > 0 \), \( f_{xx}(-1,-1) = 12 > 0 \)
\( \Rightarrow (-1,-1) \) gives the local min \( f(-1,-1) = -1 \)

Remark: Done with local max/min. How to find global max/min?
This is tricky in general. Use algebra/graph/intuition... or simply assume global max/min exists (since you are asked for it) then take
\[ \max (\text{all local max}) \Rightarrow \text{global max} \]
\[ \min (\text{all local min}) \Rightarrow \text{global min} \]