Chain Rule & Implicit Diff. (previously, we did implicit diff. separately for each individual problem. Here we have a general formula using chain rule)

- Case $F(x, y) = 0$, $y$ implicit in $x$

Think of $y$ as $y(x)$, differentiate both sides of $F(x, y) = 0$ wrt $x$.

Result: $\frac{dy}{dx} = F(x, y)

\text{Middle}: x \Rightarrow y = y(x)

\text{diff wrt } x

\frac{du}{dx} = \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx}

\text{Result:} \quad \frac{dy}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$

- Case $F(x, y, z) = 0$, $z$ implicit in $x$ & $y$

Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$

For $\frac{\partial z}{\partial x}$: take $\frac{\partial}{\partial x}$ both sides we have
Result \( u = F(x, y, z) \)

\[
\frac{\partial u}{\partial x} = \frac{\partial F}{\partial x} \left( \frac{\partial x}{\partial x} \right) + \frac{\partial F}{\partial y} \left( \frac{\partial y}{\partial x} \right) + \frac{\partial F}{\partial z} \left( \frac{\partial z}{\partial x} \right)
\]

\[\Rightarrow \frac{\partial z}{\partial x} = -\frac{F_x}{F_z}\]

Similar calculation with \( \frac{\partial}{\partial y} \), get

\[\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}\]

Eq: \( y \) implicit in \( x \) such that \( x^3 + y^3 = 6xy \)

Find \( \frac{dy}{dx} \)?
Solution: \[ x^3 + y^3 - 6xy = 0 \]

\[ F(x,y) \]

\[ \frac{dy}{dx} = - \frac{F_x}{F_y} = - \frac{3x^2 - 6y}{3y^2 - 6x} = - \frac{x^2 - 2y}{y^2 - 2x} \]

Remark:

if you do implicit diff. as before (i.e. not using the formula \[ \frac{dy}{dx} = - \frac{F_x}{F_y} \]), you should also get the same result.

Eq: the previous eq in Oct 05: \[ z \] implicit in \[ x \] and \[ y \] such that

\[ e^z = xyz + 1 - z \]

Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

Solution: \[ e^z - xyz - 1 + z = 0 \]

\[ F(x,y,z) \]

\[ \frac{\partial z}{\partial x} = - \frac{F_x}{F_z} = - \frac{-yz}{e^z - xy + 1} = \frac{yz}{e^z - xy + 1} \] (same answer as before)

\[ \frac{\partial z}{\partial y} = - \frac{F_y}{F_z} = - \frac{-xz}{e^z - xy + 1} = \frac{xz}{e^z - xy + 1} \]
* Some remarks:

- APEX only has the formula \[ \frac{dy}{dx} = - \frac{F_x}{F_y} \] (for the case \( F(x,y) = 0 \)).

- Stewart (p. 953–954) has both this formula and the formula
  \[ \frac{dz}{dx} = - \frac{F_x}{F_z}, \quad \frac{dz}{dy} = - \frac{F_y}{F_z} \] (for the case \( F(x,y,z) = 0 \)).

- Students usually confuse: in the above formula, we treat \( x,y,z \) as independent variables of \( F \). I will include an extra short note to emphasize this.

* Some hard problems using Chain Rule

I'll probably post the solution at some point, but you should try these yourselves first:

1) 2014 WT1 Problem 2
2) 2013 WT2 Problem 2(a)
3) 2012 WT1 Problem 2
§12.6 APEX Directional Derivatives & Gradient
(Also §14.6 Stewart)

Motivation:

Recall

\[ f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \]

\( \Rightarrow \) fix \( y = y_0 \), vary \( x = x_0 + h \) as \( h \to 0 \)

\( \Rightarrow \) think of this as "derivative in the direction of the x-axis"

Likewise \( f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h} \)

\( \Rightarrow \) "derivative in the direction of the y-axis"

Now: study derivative in an arbitrary direction

Formal definition:

\[ f(x, y) : \text{function in 2 variables} \]

\[ \vec{u} = <a, b> : \text{unit vector} \]

Then the directional derivative of \( f \) at \( (x_0, y_0) \) in the direction of \( \vec{u} \) is

\[ D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \]

Remark: in the above formula, \( |h| = \text{distance from } (x_0, y_0) \text{ to } (x_0 + ha, y_0 + hb) \)