Eq: \( z = e^x \sin y \), \( x = st^2 \), \( y = s^2t \)

Find \( \frac{\partial z}{\partial s} \) and \( \frac{\partial z}{\partial t} \)

Solution:

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = e^x \sin y \cdot t^2 + e^x \cos y \cdot 2st
\]

\[
= e^{st^2} \sin (s^2t) \cdot t^2 + e^{st^2} \cos (s^2t) \cdot 2st
\]

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}
\]

\[
= e^x \sin y \cdot 2st + e^x \cos y \cdot s^2
\]

\[
= e^{st^2} \sin (s^2t) \cdot 2st + e^{st^2} \cos (s^2t) \cdot s^2
\]

Result: \( z = e^x \sin y \)

Middle

\[
\begin{array}{c}
\text{Result: } z = e^x \sin y \\
\text{Middle: } x \\
y \\
s \\
t
\end{array}
\]

* The most general version of the Chain Rule:

\[
\begin{align*}
&z = f(x_1, \ldots, x_m) \text{ differentiable function in } m \text{ variables, } x_1, \ldots, x_m \\
&\text{Each } x_i \text{ is a function in } n \text{ variables } t_1, \ldots, t_n \\
&\text{for } 1 \leq i \leq m \\
\text{Then for } 1 \leq j \leq n, \text{ we have}
\end{align*}
\]

\[
\frac{\partial z}{\partial t_j} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_j} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_j} + \ldots + \frac{\partial z}{\partial x_m} \cdot \frac{\partial x_m}{\partial t_j}
\]

p.716 AP Felix, also p.951 Stewart
As before, perhaps this diagram is more illustrative.

Result: \( z = f(x_1, \ldots, x_m) \)

Middle:

\[ x_1 \quad x_2 \quad \ldots \quad x_m \]

Diff wrt: \( t_j \)

Example: Write out the chain rule for the case

\( w = f(x, y, z, t) \)

where \( x, y, z, t \) are functions of \( u \) and \( v \).

Write the chain rule for \( \frac{\partial w}{\partial u} \) and \( \frac{\partial w}{\partial v} \).

Solution:

\[
\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial u} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial u}
\]

Similarly:

\[
\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial v} + \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial v}
\]
Eg: \[ g(s, t) = f(s^2 - t^2, t^2 - s^2) \]

\[ f \text{ is differentiable, show that } g \text{ satisfies} \]

\[ t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = 0 \]

Solution: (idea: given \( s \) \& \( t \), need \( s^2 - t^2 \) and \( t^2 - s^2 \) to plug-in \( f \))

1. Find \( \frac{\partial g}{\partial s} \):
   \( g = f(x, y) \) with \( x = s^2 - t^2 \), \( y = t^2 - s^2 \)

   \[ \frac{\partial g}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = 2s \frac{\partial f}{\partial x} - 2s \frac{\partial f}{\partial y} \]

2. Find \( \frac{\partial g}{\partial t} \):
   Similarly

   \[ \frac{\partial g}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = -2t \frac{\partial f}{\partial x} + 2t \frac{\partial f}{\partial y} \]

   So \( t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = 2ts \frac{\partial f}{\partial x} - 2ts \frac{\partial f}{\partial y} + (-2st) \frac{\partial f}{\partial x} + 2st \frac{\partial f}{\partial y} \)

   \[ = 0 \]

*Eg: 2nd order partial derivatives & chain rule

⇒ 1st derivative & chain rule give (the sum of) a bunch of products

⇒ When taking 2nd der, don't forget the product rule (expect a rather messy formula)
Here's another one:

Eq: \[ z = f(r \cos \theta, r \sin \theta) \]

Use the chain rule to find \( \frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}, \frac{\partial^2 z}{\partial r^2} \)

\[
\begin{align*}
\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} = \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \\
\frac{\partial z}{\partial \theta} &= \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta} = -r \sin \theta \frac{\partial z}{\partial x} + r \cos \theta \frac{\partial z}{\partial y}
\end{align*}
\]

For \( \frac{\partial^2 z}{\partial r^2} \):

\[
\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial r} \right) = \frac{\partial}{\partial r} \left( \cos \theta \frac{\partial z}{\partial x} + \sin \theta \frac{\partial z}{\partial y} \right)
\]

We're lucky here since \( \cos \theta \) and \( \sin \theta \) don't involve \( r \); in general, need product rule.

Chain rule for thin and thin:

\[
\frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \cdot \frac{\partial y}{\partial r} = \cos \theta \frac{\partial^2 z}{\partial x^2} + \sin \theta \frac{\partial^2 z}{\partial y \partial x}
\]

Similarly:

\[
\frac{\partial}{\partial r} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) \cdot \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) \cdot \frac{\partial y}{\partial r} = \cos \theta \frac{\partial^2 z}{\partial x \partial y} + \sin \theta \frac{\partial^2 z}{\partial y^2}
\]

Overall:

\[
\frac{\partial^2 z}{\partial r^2} = \cos \theta \left( \cos \theta \frac{\partial^2 z}{\partial x^2} + \sin \theta \frac{\partial^2 z}{\partial y \partial x} \right) + \sin \theta \left( \cos \theta \frac{\partial^2 z}{\partial x \partial y} + \sin \theta \frac{\partial^2 z}{\partial y^2} \right)
\]

This answer is good enough, or you can use Clairaut's theorem, get \( \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} \)

and simplify:

\[
\frac{\partial^2 z}{\partial r^2} = \cos^2 \theta \frac{\partial^2 z}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 z}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 z}{\partial y^2}
\]