P(x,y,z) has spherical coordinates
(ρ, θ, φ) with the following
* ρ = distance to the origin = |OP|
* φ = angle from Oz to OP
(non-oriented: 0 ≤ φ ≤ π) so that

\[ z = ρ \cos φ \]

Keep in mind: 0 ≤ φ ≤ π (not 2π)

Length of OP': |OP'| = ρ sin φ

Polar coordinate of P' on xy-plane with r = ρ sin φ
\[ x = ρ \sin φ \cos θ \]
\[ y = ρ \sin φ \sin θ \]
with 0 ≤ θ ≤ 2π

Summary:
\[ x = ρ \sin φ \cos θ \]
\[ y = ρ \sin φ \sin θ \]
\[ z = ρ \cos φ \]
\[ 0 ≤ ρ, \ 0 ≤ φ ≤ π, \ 0 ≤ θ ≤ 2π \]
Some examples:

1) Show Eq. 1, p. 1058 on how to draw the point with spherical coordinate \((2, \frac{\pi}{4}, \frac{\pi}{3})\) ("order" \((\rho, \theta, \phi)\))

2) Some solids & surfaces in spherical coordinates:

a) \(\rho = 3\) \(\Rightarrow\) sphere of radius 3, center origin

b) \(0 \leq \rho \leq 3\)
   (together with the default inequalities, \(0 \leq \phi \leq \pi\), \(0 \leq \theta \leq 2\pi\)) \(\Rightarrow\) the solid sphere of radius 3, center origin

c) \(0 \leq \rho \leq 3\)
   \(0 \leq \phi \leq \frac{\pi}{2}\)
   (the default \(0 \leq \theta \leq 2\pi\)) \(\Rightarrow\) upper half of the solid sphere in part (b).
   (either use picture or use that \(\cos \phi > 0\) when \(0 \leq \phi \leq \frac{\pi}{2}\) to have that \(z > 0\))

d) Represent the \(\frac{1}{8}\) of the solid sphere of radius \(a\) centered at \(0\) with the conditions: \(x \leq 0\), \(y \geq 0\), \(z \geq 0\) in spherical coordinates:
   \(0 \leq \rho \leq a\)
   \(x \leq 0, y \geq 0\) \(\Rightarrow\)
   \(\frac{\pi}{2} \leq \theta \leq \pi\)
   \(z \geq 0\): either use picture or solve for \(\cos \phi > 0\): \(0 \leq \phi \leq \frac{\pi}{2}\)
Some useful identities:

\[ x^2 + y^2 + z^2 = \rho^2 \quad \rho \sqrt{x^2 + y^2} = \rho \sin \phi \]

\[ \tan \theta = \frac{y}{x} \quad \cos \phi = \frac{z}{\rho} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \]

\[ SS S \text{ in Spherical Coordinates.} \]

\* General rule: change every \( x, y, z \) to \( \rho, \theta, \phi \)

\- Don't forget the extra factor \( \rho^2 \sin \phi \)

\* Formula:

Solid \( E \) in spherical coordinates are described by

\[ a \leq \rho \leq b \quad ; \quad c \leq \phi \leq d \]

\[ \alpha \leq \theta \leq \beta \]

(Show figure 7 in p. 1058)

\[ \iiint_E f(x,y,z) \, dV = \iiint_{c \times a} \int_{\alpha}^{\beta} \int_{c}^{b} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]

Note: since \( a, b, c, d, \alpha, \beta \) are constants; doesn't matter which order of \( \int \int \int \) that we use.

Popular choices are \( d\rho d\theta d\phi \) as in the textbook (Stewart)

or \( d\rho d\phi d\theta \) (as in some other books)
* A bit more general:

E in spherical \((\rho, \theta, \phi)\) has the form

\[ a \leq \theta \leq \rho, \quad c \leq \phi \leq d \]

\[ g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi) \]

Then

\[
\iiint_E f(x, y, z)\,dV = \iiint_{c \leq \rho \leq g_2(\theta, \phi)} f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \,d\rho \,d\theta \,d\phi
\]

**Eq:** B is the \( \frac{1}{8} \) of the solid unit ball center 0 with \( x \leq 0, y \leq 0, z \geq 0 \)

Given \( \iiint_{B} e^{(x^2 + y^2 + z^2)^{3/2}} \,dV \)

a) Write \( \iiint_{B} \) in \( \iiint_{\cdots} dz \,dy \,dx \)

b) Write \( \iiint_{B} \) in cylindrical coordinate

c) Write \( \iiint_{B} \) in spherical coordinate \( \text{and evaluate} \) (yes, you can)
a) \(0 \leq z \) and \(z^2 + y^2 + z^2 \leq 1\), hence \(0 \leq z \leq \sqrt{1-x^2-y^2}\) 

\[
\iiint_D e^{(x^2 + y^2 + z^2)^{3/2}} \, dz \, dA
\]

\[
\int_{-1}^0 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2-y^2}} \int_0^{\sqrt{1-x^2-y^2}} e^{(x^2 + y^2 + z^2)^{3/2}} \, dz \, dy \, dx
\]

(how? write the quarter-circle as graph \(y = -\sqrt{1-x^2}\) when \(-1 \leq x \leq 0\))

b) Cylindrical: \(0 \leq z \leq \sqrt{1-r^2}\)  

\(D \) in \((r, \theta)\):  
\[
\begin{cases} 
\pi \leq \theta \leq \frac{3\pi}{2} \\
0 \leq r \leq 1 
\end{cases}
\]

Answer:  
\[
\int_{\pi/2}^{3\pi/2} \int_0^1 \int_0^{\sqrt{1-r^2}} e^{(r^2 + z^2)^{3/2}} r \, dz \, dr \, d\theta
\]

c) Describe the \(\frac{1}{8}\) of the sphere in spherical coord.
\[
0 \leq \rho \leq 1
\]

Since \(x \leq 0, y \leq 0\) \rightarrow III quadrant \rightarrow \pi \leq \theta \leq \frac{3\pi}{2}\]
(Extra page for the long solution of the current example)

Since \( z \geq 0 \Rightarrow 0 \leq \phi \leq \frac{\pi}{2} \)

So, rewrite the integral in spherical coordinate:

\[
\int_0^\frac{\pi}{2} \int_0^{\frac{3\pi}{2}} \int_0^1 e^{-r^2}\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\]

\[
= \int_0^\frac{\pi}{2} \int_0^{\frac{3\pi}{2}} \frac{1}{3} \sin \phi \cdot e^3 \Bigg|_0^1 \, d\theta \, d\phi
\]

\[
= \int_0^\frac{\pi}{2} \int_0^{\frac{3\pi}{2}} \frac{1}{3} (e - 1) \sin \phi \, d\theta \, d\phi
\]

\[
= \int_0^\frac{\pi}{2} \frac{\pi}{6} (e - 1) \sin \phi \, d\phi = \frac{\pi}{6} (e - 1).
\]