So, \( E = \{(x,y,z) \mid (x,y) \in D, \; u_1(x,y) \leq z \leq u_2(x,y)\} \)

\[ \Rightarrow \text{show Figure 2 in p. 1042 Stewart} \]

**Formula:** (do \( dz \) first)

\[ \iiint_E f(x,y,z)\,dV = \iint_D (\int_{u_1(x,y)}^{u_2(x,y)} f \,dz)\,dA \]

(then use §15.3 to handle \( \iint_D \cdots\,dA \) (move vertical/horizontal lines))

* **Type 2:** similarly:

  \( x \) between graphs of 2 functions in \( y,z \) with domain \( D \)

  \[ E = \{(x,y,z) \mid (y,z) \in D, \; u_1(y,z) \leq x \leq u_2(y,z)\} \]

  \[ \Rightarrow \text{show Figure 7 in p. 1044 Stewart} \]

**Formula:** (do \( dx \) first)

\[ \iiint_E f(x,y,z)\,dV = \iint_D (\int_{u_1(y,z)}^{u_2(y,z)} f \,dx)\,dA \]

* **Type 3:** similarly

  \( y \) between graphs of 2 functions in \( x,z \) with domain \( D \).
\[
E = \{ (x,y,z) \mid (x,z) \in D, \; u_1(x,z) \leq y \leq u_2(x,z) \}
\]

\[\Rightarrow\] Figure 8 in p. 1044 Stewart

Formula: (do \(dy\) first)

\[
\iiint_E f(x,y,z) \, dV = \iiint_D \left( \int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z) \, dy \right) \, dA
\]

Remarks:
- Forget the name type 1, type 2, type 3. The point is.
  - express one variable between graphs of 2 functions in the
  other 2 variables, take \(\int \ldots \, d(\text{thin one variable})\), then

\[
\iiint_{\text{domain of 2 functions}} \ldots \, dA
\]

- How to use the above formulas:
  (a) Use picture + lines parallel to \(z\)-axis (or \(x\)-axis,)
  (b) From given equations \(\Rightarrow\) figure out the inequalities
      describing your solid
  (c) Usually (a) + (b) + lots of practice.
- Inequalities: in $\mathbb{R}^3$, a surface of the form $\star = \star$
  (eg: $y = x^2 + z^2$, $x^2 + y^2 + z^2 = 3$, $\ldots$)

breaks the 3-d spaces in 2 pieces. One piece is given by $\star \leq \star$
other piece by $\star \geq \star$

⇒ plug-in a random point to check if $\leq$ or $\geq$ holds

Eg(a) Find $\iiint E \sqrt{x^2 + z^2} \, dV$

where $E$ is bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$

(Hint: do $dy$ first)

b) Try to do $dx$ first & identify the difficulties.

Solution: Try to sketch the surfaces $y = x^2 + z^2 \& y = 4$ very roughly & imagine your solid $E$

⇒ \{ "Inside" the paraboloid $y = x^2 + z^2$

\quad $\Rightarrow$ \{ $x^2 + z^2 \leq y$

\quad $y \Rightarrow$ \{ "On the left" of the plane $y = 4$

Overall: $x^2 + z^2 \leq y \leq 4$ \quad Need domain $D$

$u_1(x,z) \quad u_2(x,z)$ in $(x,z)$
* For $D$ in $(x, z)$, either you can project the picture to the $xz$-plane, OR notice that from $x^2 + z^2 \leq y \leq 4$, get the inequality: \( x^2 + z^2 \leq 4 \) so that we can solve for $y$.

So:

\[
\iiint_D \left( \frac{\sqrt{x^2 + z^2}}{\sqrt{x^2 + z^2}} \right) \, dA
\]

\[
= \int_0^{2\pi} \int_0^2 r (4 - r^2) \, r \, dr \, d\theta \]

\[
= \int_0^{2\pi} \left( \frac{4r^3}{3} - \frac{r^5}{5} \right) \bigg|_0^2 \, d\theta = 2\pi \left( \frac{32}{3} - \frac{32}{5} \right)
\]

b) If doing $dx$ first: from the last inequalities:

\[
\begin{align*}
\{ x^2 + z^2 \leq y & \text{ need } x \leq \? \text{ Note: } x^2 \leq y - z^2 \\
y \leq 4 & \text{ answer: } -\sqrt{y - z^2} \leq x \leq \sqrt{y - z^2}
\end{align*}
\]

Domain $D_1$ in $(y, z)$? We have $y \leq 4$, but there's an extra inequality to solve for $x$, need $\sqrt{y - z^2}$ makes sense. So $D_1$ is $z$.

You can also try to project to $yz$-plane to get $D_1$.

\[
\iiint_{D_1} \left( \frac{\sqrt{y - z^2}}{\sqrt{x^2 + z^2}} \right) \, dx \, dA
\]

HARD to calculate!!!
Eq: Evaluate \( \iiint_E z \, dV \) where \( E \) is the solid tetrahedron bounded by the 4 planes \( x = 0, \ y = 0, \ z = 0, \) and \( x + y + z = 1 \)

(Hint: roughly sketch \( E \), express one variable between the 2 functions in other 2 variables, also identify the domain \( D \) in those 2 variables)

Describe the solid.

We have:

\[ x > 0, \ y > 0, \ z > 0, \ x + y + z \leq 1 \] (since below this plane)

\[ 0 \leq z \leq 1 - x - y \]

(Why? solid above plane \( z = 0 \), so \( z \geq 0 \)

below plane \( x + y + z = 1 \) so \( x + y + z \leq 1 \), or \( z \leq 1 - x - y \))

How to get \( D \)? Either use picture project to the xy-plane

Or use inequalities:

\( x \geq 0, \ y \geq 0 \) AND \( y = 1 - x \)

The EXTRA: \( 0 \leq 1 - x - y \) from (*)

So:

\[ \iiint_D \left( \int_0^{1-x-y} z \, dz \right) \, dA = \iiint_D (1-x-y)^2 \, dA \]

\[ = \int_0^1 \int_0^{1-x} \left( \frac{y + x - 1}{2} \right) dy \, dx = \int_0^1 \left( \frac{y + x - 1}{6} \right)^3 \, dx \]

\[ = \int_0^1 \frac{(x-1)^3}{6} \, dx = \int_0^1 \frac{(x-1)^4}{24} \, dx = \frac{1}{24} \]

Eq: \[ \int_0^1 \int_0^{x^2} \int_0^{y} f(x, y, z) \, dz \, dy \, dx \]

\[ = \int_?^? \int_?^? \int_?^? f(x, y, z) \, dx \, dz \, dy \]

Describe E:
\[
\begin{align*}
0 &\leq z \leq y \quad (1) \\
0 &\leq y \leq x^2 \quad (2) \\
0 &\leq x \leq 1 \quad (3)
\end{align*}
\]

To do $dx \, dz \, dy$, need $? \leq x \leq ?$

Get: $x \leq 1$ from (3)

Get $\sqrt{y} \leq x$ from (2) and $0 \leq x$ from (3) $\Rightarrow \sqrt{y} \leq x$

Overall: $\sqrt{y} \leq x \leq 1$

Domain $D$ in $(y, z)$:
\[
\begin{align*}
0 &\leq z \leq y \\
0 &\leq y \\
\text{AND the extra $\sqrt{y} \leq 1$ so that we can solve for $x$ from $\sqrt{y} \leq x \leq 1$}
\end{align*}
\]

Answer:
\[
\iiint_D \int_0^1 f(x, y, z) \, dx \, dA
\]

\[= \int_0^1 \int_0^{\sqrt{y}} \int_0^{y} f(x, y, z) \, dx \, dz \, dy\]