\[ = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1 + \cos 4\theta}{4} \, d\theta = \frac{1}{4} \left( \theta + \frac{\sin 4\theta}{4} \right) \bigg|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{8} \]

Trick to find \(\alpha, \beta, h_1(\theta), h_2(\theta)\) (similar to "moving lines", we are)

- Rotate the rays from 0 that cut in a line segment
- First ray like that \(\Rightarrow \alpha\)
- Last ray \(\Rightarrow \beta\)
- For each line segment: closest point \(\Rightarrow h_1(\theta)\)
- Furthest point \(\Rightarrow h_2(\theta)\)

"Default" inequalities for \(\theta\): \(0 \leq \theta \leq 2\pi\), BUT everything remains true if replacing \([0, 2\pi]\) by an interval of length \(2\pi\) 

\(\Rightarrow \alpha, \beta\) need not be in \([0, 2\pi]\), any choice with 

\[0 \leq \beta - \alpha \leq 2\pi: \text{OK.}\]

Eg: Find volume of the solid under \(z = x^2 + y^2\), above the \(xy\)-plane & inside the cylinder \(x^2 + y^2 = 2x\)

(Hint: figure out \(D\) = domain to take \(\iiint_D (x^2 + y^2) \, dA\)

On the \(xy\)-plane, describe the curve \(x^2 + y^2 = 2x\)

\(\Rightarrow\) describe \(D\) in polar coordinates.
Answer:
On the $xy$-plane, the curve $x^2 + y^2 = 2x$ is the circle $x^2 - 2x + 1 + y^2 = 1$,
$(x - 1)^2 + y^2 = 1$ (center $(1,0)$).
So $D$ is the inside of this circle.

In polar coordinates, this circle is:

$$\frac{x^2 + y^2}{r^2} = \frac{2x}{r^2} = 2 \cos \theta$$

so $r = 2 \cos \theta$.

Description of $D$ in polar coordinates:

$$\left\{ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \right\}$$

$$0 \leq r \leq 2 \cos \theta$$

Vol = $\iint_D (x^2 + y^2) \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2 \cos \theta} r^2 \cdot r \, dr \, d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ \frac{r^4}{4} \right]_0^{2 \cos \theta} \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4 \cos^4 \theta \, d\theta$$

(use: $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$)

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos 2\theta)^2 \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + 2 \cos 2\theta + \cos^2 2\theta) \, d\theta$$

(again: $\cos^2 2\theta = \frac{1 + \cos 4\theta}{2}$)

$$= (\theta + \sin 2\theta)\bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 4\theta}{2} \, d\theta$$

$$= \pi + \frac{1}{2} \left( \theta + \frac{\sin 4\theta}{4} \right)\bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{3\pi}{2}$$
Leave this as homework (this is actually in the final) for 2014.

In the $xy$-plane, the disk $x^2 + y^2 \leq 2x$ is cut into 2 pieces by the line $y = x$. Let $D$ be the larger piece.

Find the volume of the solid below $z = \sqrt{x^2 + y^2}$ and above $D$.

**Solution:** as before, by completing the square, get

\[
x^2 - 2x + 1 + y^2 \leq 1
\]
\[(x - 1)^2 + y^2 \leq 1
\]

$\Rightarrow$ inside the disk center $(1,0)$ radius 1.

Using polar coordinates, the given inequality becomes

$x^2 + y^2 \leq 2x$ becomes $r^2 \leq 2r\cos\theta$, hence $r \leq 2\cos\theta$.

Description of $D$ in polar coordinates: $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{4}$, $0 \leq r \leq 2\cos\theta$.

Now

\[
\text{Vol} = \iint_D \sqrt{x^2 + y^2} \, dA = \int_{-\pi/2}^{\pi/4} \int_0^{2\cos\theta} r \, r \, dr \, d\theta
\]

\[
= \int_{-\pi/2}^{\pi/4} \frac{8\cos^3\theta}{3} \, d\theta \quad \text{(keep one factor $\cos\theta$, write $\cos^2\theta = 1 - \sin^2\theta$)}
\]

\[
= \frac{8}{3} \int_{-\pi/2}^{\pi/4} \cos\theta (1 - \sin^2\theta) \, d\theta
\]

\[
= \frac{8}{3} \left[ \sin\theta - \frac{\sin^3\theta}{3} \right]_{-\pi/2}^{\pi/4} = \frac{2}{9} \left( 5\sqrt{2} + 8 \right)
\]
Stewart §15.5 Application of SS
(or APEX §13.4)

Only cover: mass, moments, center of mass
(exactly APEX §13.4 or the first part of Stewart §15.5)

Skip: other topics in Stewart §15.5
...will also skip Surface Area in Stewart §15.6
APEX §13.5

* Mass vs density:

Slogan: density = derivative of mass
mass = \int \text{of density}

Recall Calculus of 1-var functions:
Previously in calculus of 1 variable:

On a 1-dimensional object (e.g., "interval \([a, b]\)"

or a "thin stick")

\[
density \, \rho = \frac{d(mass)}{dt}
\]

or:

\[
\rho = \frac{dm}{dt}
\]

or:

\[
m = \int \rho \, dt
\]

Now in calculus of 2-var:

2-dimensional region \(D\) (such as a lamina)

\[
density \, \rho = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A} = \frac{dm}{dA}
\]

So:

\[
m = \iint_D \rho(x, y) \, dA
\]

Similar situation: electric charge distributed on \(D\) with

density \(\sigma(x, y)\) then

\[
\text{total charge} = \iint_D \sigma(x, y) \, dA
\]
Electronic Charge distributed over $D$ with density 

$$\rho(x,y) = xy$$

Find the total charge 

$$\text{total charge} = \iiint_D \rho(x,y)\,dA$$

$$= \iint_D xy\,dA = \int_0^1 \int_{1-x}^1 xy\,dy\,dx = \int_0^1 \frac{xy^2}{2} \bigg|_{1-x}^1 \,dx$$

$$= \int_0^1 \frac{x}{2} \left(1 - (1-x)^2\right)\,dx = \int_0^1 \left(x - \frac{x^3}{2}\right)\,dx = \left(\frac{x^3}{3} - \frac{x^4}{8}\right) \bigg|_0^1$$

$$= \frac{5}{24}$$

---

**Moments & Center of Mass**: consider a lamina that occupies a region $D$ & has density $\rho(x,y)$

**Define**: moment = product of mass & directed distance from the axis.
(2 axes $\Rightarrow$ 2 moments

directed distance $=$ coordinate (could be < 0)

e.g. directed distance from x-axis $=$ y-coordinate

directed distance from y-axis $=$ x-coordinate

Formula: moment about x-axis

$$M_x \approx \sum_{i=1}^{m} \sum_{j=1}^{n} y_i^+ \Delta m_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} y_i^+ p(x_i^*, y_i^*) \Delta A$$

So

$$M_x = \iint_D y \rho(x, y) \, dA$$

Similarly, moment about y-axis

$$M_y = \iint_D x \rho(x, y) \, dA$$

Center of mass (show picture p. 1029):

is the coordinate $(\bar{x}, \bar{y})$ given by

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x \rho(x, y) \, dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y \rho(x, y) \, dA$$

where we recall $m = \iint_D \rho(x, y) \, dA$
Quick remark: beware how these notations "mismatch", the point is:
- $M_x \leftrightarrow$ distance from y-axis
- $M_y \leftrightarrow$ distance from x-axis

Read eg 2 p. 1029 Stewart

Another eg: semicircular lamina of radius $a$ centered at the origin. Density at any point is proportional to distance from the center. Find center of mass.

\[
\begin{aligned}
\text{(need to: figure out } p(x,y), \quad D, \quad m, \quad M_x, \quad M_y) \\
\Rightarrow \bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}
\end{aligned}
\]

\[
p(x,y) = kr \sqrt{x^2 + y^2}
\]

where $k$ is a constant

\[
m = \iint_D p(x,y) \, dA = \iint_D kr \sqrt{x^2 + y^2} \, dA
\]

Use polar coordinate description of $D$: $0 \leq \theta \leq \pi$, $0 \leq r \leq a$

So

\[
m = \int_0^\pi \int_0^a kr^3 \, r \, dr \, d\theta = \int_0^\pi kr^3 \left| \int_0^a r \, dr \right| d\theta = \frac{ka^3}{3} \pi
\]

\[
\bar{x} = \frac{1}{m} \iint_D x \, p(x,y) \, dA = \frac{1}{m} \int_0^\pi \int_0^a r \cos \theta \, kr \, r \, dr \, d\theta
\]

\[
= \frac{1}{m} \int_0^\pi k \cos \theta \frac{a^4}{4} \, d\theta = \frac{ka^4}{4m} \sin \theta \bigg|_0^\pi = 0
\]
(By observing that $D$ is symmetric and $\rho(x,y)$ is also symmetric with respect to the $y$-axis, you know that the center of mass lies in the $y$-axis, hence $\bar{x} = 0$ without doing the above calculation.

\[
\bar{y} = \frac{1}{m} \iint_D y\rho(x,y)\,dA = \frac{1}{m} \int_0^\pi \int_0^a r\sin\theta kr\cdot r\,dr\,d\theta
\]

\[
= \frac{1}{m} \int_0^\pi \frac{k\sin\theta}{4} a^4 d\theta = \frac{ka^4}{4m} \left[ (-\cos\theta) \right]_0^\pi = \frac{2ka^4}{4m}
\]

\[
= \frac{2ka^4}{4\cdot ka^3 \frac{\pi}{3}} = \frac{3a}{2\pi} \quad \text{answer: (0, } \frac{3a}{2\pi})
\]

Another eg: (leave as homework if not enough time)

Same as last eg, this time density is inversely proportional to distance from the $\underline{\text{origin}}$.

Solution: similar to previous solution of the previous eg., this time

$\rho(x,y) = \frac{k}{\sqrt{x^2 + y^2}}$ where $k$ is a constant

$D$ in polar coordinates:

$0 \leq r \leq a$, $0 \leq \theta \leq \pi$

$m = \iint_D \rho(x,y)\,dA = \int_0^\pi \int_0^a \frac{k}{r} r\,dr\,d\theta = \int_0^\pi kr \bigg|_0^a \,d\theta$

\[
= \int_0^\pi ka \,d\theta = ka\pi
\]