Case 2: D in type II

- y between horizontal lines $y = c \& y = d$
- x between curves $x = h_1(y) \& x = h_2(y)$

**Example**

- Find the volume of the solid obtained by rotating the region bounded by $y = h_1(x)$ and $y = h_2(x)$ about the y-axis.

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For a region not of type II,

- Test: Check if any horizontal line cut in more than 1 line segment.

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- Cut in 2 line segments.
Formula:
\[ D = \{(x,y) \mid c \leq y \leq d, \ h_1(y) \leq x \leq h_2(y)\} \]
\[ \iint_D f \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) \, dx \, dy \]

General method (similar to type I) when using type II:
- Sketch D, make sure type II
- Move horizontal lines
  * first to touch \( \Rightarrow c \)
  * last \( \Rightarrow d \)
  (usually involve points of intersection)
  (could happen: \( h_1 + h_2 \); more)
- leftmost \( \Rightarrow h_1(y) \)
  rightmost \( \Rightarrow h_2(y) \)
  (then 1 formulas
  \( \Rightarrow \) need to break the integral)

\( \text{Eq:} \)
\( \text{Dec:} \)
\( \text{Pd:} \)
Eq. (when \( g_1, g_2 \) or \( h_1, h_2 \) given by more than 1 formulas)

\[
P(x)
\]
\[
Q(x)
\]
\[
a \quad \chi \quad b
\]

Yes: type I, write down formula for \( \int_D f(x,y) \, dA \)?

So \( g_1 \) is actually

\[
g_1(x) = \begin{cases} 
P(x) & \text{if } a \leq x \leq \chi \\ 
Q(x) & \text{if } \chi \leq x \leq b 
\end{cases}
\]

So \( g_2 \) is actually

\[
g_2(x) = \begin{cases} 
Q(x) & \text{if } a \leq x \leq \chi \\ 
P(x) & \text{if } \chi \leq x \leq b 
\end{cases}
\]

Answer:

\[
\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) \, dy \, dx = \int_a^\chi \int_{P(x)}^{Q(x)} f(x,y) \, dy \, dx + \int_\chi^b \int_{Q(x)}^{P(x)} f(x,y) \, dy \, dx
\]

Eq: Find \( \iint_D xy \, dA \) where \( D \) is bounded by the line

\[y = x - 1 \quad \& \quad \text{parabola } \quad y^2 = 2x + 6\]

Using 2 methods: (note: draw \( D \) & see \( D \) is of type I & II)

a) Regard \( D \) as type I.

b) Regard \( D \) as type II.
Points of intersection:

Solve \((x-1)^2 = 2x + 6\)

\[x^2 - 4x - 5 = 0\]

\((x = -1)\) or \((x = 5)\)

\((y = -2)\) \hspace{1cm} \((y = 4)\)

a) Formula for type I: \(g_2\)

\[g_1 \Rightarrow g_2(x) = \begin{cases} -\sqrt{2x+6} & \text{if } -3 \leq x \leq -1 \\ x - 1 & \text{if } -1 \leq x \leq 5 \end{cases}\]

So:

\[
\int_{-3}^{-1} \int_{\sqrt{2x+6}}^{-\sqrt{2x+6}} xy \, dy \, dx + \int_{-1}^{5} \int_{x-1}^{\sqrt{2x+6}} xy \, dy \, dx
\]

\[
= \int_{-3}^{-1} \left( x \left( \frac{y^2}{2} \right) \bigg|_{\sqrt{2x+6}}^{-\sqrt{2x+6}} \right) dx + \int_{-1}^{5} \left( x \left( \frac{y^2}{2} \right) \bigg|_{x-1}^{\sqrt{2x+6}} \right) dx
\]

\[
= \int_{-3}^{-1} \frac{x}{2} \left( 2x+6 - (2x+6) \right) \, dx + \int_{-1}^{5} \frac{x}{2} (2x+6) \, dx - \int_{-1}^{5} x^3 \, dx
\]

\[
= \left[ -\frac{x^3}{6} + 2x^2 + \frac{5x}{4} \right]_{-3}^{-1} = 36
\]
b) Now if we use type II:

\[ h_1: \quad \text{so} \quad x = h_1(y) = \frac{y^2 - 6}{2} \]

\[ h_2: \quad \text{so} \quad x = h_2(y) = y + 1 \]

\[
\int_{-2}^{4} \int_{\frac{y^2 - 6}{2}}^{y + 1} xy \, dx \, dy = \int_{-2}^{4} \left( y \frac{x^2}{2} \bigg|_{\frac{y^2 - 6}{2}}^{y + 1} \right) \, dy
\]

\[
= \int_{-2}^{4} \frac{y}{2} \left( (y + 1)^2 - \left( \frac{y^2 - 6}{2} \right)^2 \right) \, dy = \frac{1}{2} \int_{-2}^{4} \left( -\frac{y^5}{4} + 4y^2 + 2y - 8 \right) \, dy
\]

\[
= \frac{1}{2} \left( -\frac{y^6}{24} + \frac{4}{3} y^3 + y^2 - 8y \right) \bigg|_{-2}^{4}
\]

\[ = 36 \]
So, remark: when \(D\) is of both type, the choice of formulas & say, do dydx (type I) or dx dy (type II) may simplify the problem.

Eg: \[ \int_0^1 \int_x^1 \sin(y^2) \, dy \, dx = ? \]

(Hint: draw \(D\) & switch to dx dy)

It's hard (perhaps "impossible") to integrate \(\sin(y^2)\) with respect to \(y\), change to dx dy as follows:

Sketch \(D\):

![Graph of \(D\) with bounds 0 to 1 for both variables.]

Use formula for type II: \[ \int_0^1 \int_0^y \sin(y^2) \, dx \, dy \]

\[ = \left[ \int_0^1 \sin(y^2) \cdot x \right]_0^y \, dy = \int_0^1 y \sin(y^2) \, dy \quad (u = y^2) \]

\[ = \int_0^1 \sin(u) \cdot \frac{du}{2} = \frac{-\cos u}{2} \bigg|_0^1 = \frac{-\cos 1}{2} + \frac{1}{2} \]

(Serious mistake if you conveniently switch to \(\int_0^1 \int_0^1 \sin(y^2) \, dx \, dy\))
3) **General case:** $D$ neither type I nor II

Property: $D = D_1 \cup D_2$ : no overlap other than boundary

then $\iiint f(x,y) \, dA = \iint_{D_1} f \, dA + \iint_{D_2} f \, dA$

**Idea:** break $D$

**eg:**

![Diagram of a region $D$ divided into two subregions $D_1$ and $D_2$.]

**How to break?**

One possibility:

- $D_1$: type I
- $D_2$: type II

(Other possibilities: 3 regions of type I, 3 regions of type II)

**Using $\iint$ to calculate area:**

$\text{Area} (D) = \iint_D 1 \, dA$

Why? Isn't this supposed to be some volume??!!?

Explain: Yes, volume of \underline{solid} with base $D$ & constant height 1

$\Rightarrow$ Height $\times$ Base area $= \text{Area} (D)$

constant height $= 1$