Eq: Evaluate \( \int_1^3 \int_1^5 \frac{\ln y}{xy} \, dy \, dx \) in 2 ways.

\[ \text{Doing } dy \, dx: \quad \int_1^3 \frac{1}{x} \left( \frac{\ln y}{2} \right)^5 \left( y \, du = \frac{1}{y} dy \right) \left( \text{simplify: } u = \frac{\ln y}{2} \right) \]
\[ = \int_1^3 \left( \frac{\ln 5}{2} \right)^2 \cdot \frac{1}{x} \, dx = \frac{\left( \frac{\ln 5}{2} \right)^2}{2} \ln x \bigg|_1^3 = \frac{\left( \frac{\ln 5}{2} \right)^2}{2} \ln 3 \]

\[ \text{Doing } dx \, dy: \quad \int_1^5 \int_1^3 \frac{\ln y}{xy} \, dx \, dy = \int_1^5 \frac{\ln y}{y} \ln x \bigg|_1^3 \, dy \]
\[ = \int_1^5 \ln 3 \cdot \frac{\ln y}{y} \quad (\text{u-subst } u = \ln y \quad \text{as above}) \]
\[ = \ln 3 \cdot \left( \frac{\ln y}{2} \right)^2 \bigg|_1^5 = \ln 3 \cdot \frac{\left( \frac{\ln 5}{2} \right)^2}{2} \]

Eq: Find volume of the solid \( S \) bounded by the paraboloid \( x^2 + 2y^2 + z = 16 \), the planes \( x = 2, \ y = 2 \), and the 3 coordinate planes.

Solution (you'll see this type of question many times. Usually the difficulty is not about evaluating the integral, but to figure out the right integral to evaluate).

This paraboloid is the graph: \( z = 16 - x^2 - 2y^2 \) \( f(x,y) \)

From the planes \( x = 2, \ y = 2 \), 3 coordinate planes \( \Rightarrow \) domain \( R \) of \( f(x,y) \) should be \( [0,2] \times [0,2] \) (how? Try to imagine what these planes look like in here).
\[ \text{Vol} = \iint_R f \, dA = \int_0^2 \int_0^2 (16 - x^2 - 2y^2) \, dx \, dy = \int_0^2 (16x - \frac{x^3}{3} - 2y^2) \bigg|_0^2 \, dy \\
= \int_0^2 (32 - \frac{8}{3} - 4y^2) \, dy = \left( \frac{88}{3} y - \frac{4}{3} y^3 \right) \bigg|_0^2 = \frac{176}{3} - \frac{32}{3} \\
= 48 \]

**Eq:** Find \( \int_0^2 \int_0^3 ye^{-xy} \, dy \, dx \) by changing the order \( dy \, dx \rightarrow dx \, dy \) first (which makes the problem simpler)

**Sol:** \( \int_0^3 \int_0^2 ye^{-xy} \, dx \, dy = \int_0^3 (y \cdot e^{-xy}) \bigg|_0^2 \, dy \\
= \int_0^3 (-e^{-2y} + e^0) \, dy \\
= \left( -\frac{e^{-2y}}{-2} + y \right) \bigg|_0^3 = \frac{e^{-6}}{-2} + 3 - \frac{1}{2} = \frac{e^{-6} + 5}{2} \)

**Eq:** \( R = \{(r, \theta) : 0 \leq r \leq 2, \ 0 \leq \theta \leq \pi \} \)

Find \( \iint_R r \sin^2 \theta \, dA \)
\[
\int_0^{\pi} \int_0^{2} r \sin^2 \theta \, dr \, d\theta = \int_0^{\pi} \left( \sin^2 \theta \cdot \frac{r^2}{2} \right) \bigg|_0^2 \, d\theta = \int_0^{\pi} 2 \sin^2 \theta \, d\theta
\]

\[
= \int_0^{\pi} (1 - \cos 2\theta) \, d\theta \\
= \left( \theta - \frac{\sin 2\theta}{2} \right) \bigg|_0^{\pi} \\
= \pi
\]

§ 15.3 Double & Iterated Integrals over General Domain (instead of rectangles)

• What we have learned: \( R = \text{rectangle} \)
  1) Def of \( \iint_R f \, dA \) using Riemann sums.
  2)* Change \( \iint_R f \, dA \) to iterated \( \int \int \) & calculate

• Now in § 15.3: general domain \( D \)
  1) Theoretical def of \( \iint_D f \, dA \) (not really important)
    - Find rectangle \( R \) containing \( D \)
    - Define \( F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ in } D \\ 0 & \text{if } (x,y) \text{ in } R \text{ but not in } D \end{cases} \)
  2) Define \( \iiint_D f \, dA = \iint_R F \, dA \)
2)*** Change $\iint_D f \, dA$ to iterated $\int \int_D \, dA$ and calculate

Rule: 
- the outside integral $\int_D \, dA$ always numbers
- the inside integral $\int \, dA$ could be functions in the other variable
- switching $dx \, dy \leftrightarrow dy \, dx$ & $\int \, \int_D \, dA \leftrightarrow \int \, \int_D \, dA$: more complicated & not always possible using one integral, see later

Case 1: $D$ has type I

$x$ between 2 vertical lines

$y$ between 2 functions in $x$

eg:

\[0 \leq a < b\]

\[y \uparrow\]

\[g_1(x) \leq y \leq g_2(x)\]

\[x \downarrow\]

\[0 \leq x \leq b\]

\[y \uparrow\]

\[g_f(x) \leq y \leq g_l(x)\]

\[x \downarrow\]

\[0 \leq x \leq b\]
eg: **NOT** type I:

![Diagram showing a vertical line cutting a region into 2 segments]

**Explain:** When some "moving vertical line" cut in more than 1 segment

**Formula:** $D$ of type I

$$D = \{(x,y) ; a \leq x \leq b , g_1(x) \leq y \leq g_2(x)\}$$

$$\iint_D f \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f \, dy \, dx$$

**Eq:** Evaluate $\iint_D (x+2y) \, dA$, $D$: region bounded by $y=2x^2$ and $y=1+x^2$

**General method when using type I:**
- Sketch $D$, make sure it's type I
- Moving vertical lines:
  - First vertical line to touch $D$ $\Rightarrow a$ (usually involve points of intersection)
  - Last vertical line $\Rightarrow b$
  - Lowest point $\Rightarrow g_1(x)$ (could happen that $g_1$ or/and $g_2$)
  - Highest point $\Rightarrow g_2(x)$ (need more than 1 formula $\Rightarrow$ break the integral)
Find points of intersection: set $2x^2 = 1 + x^2$

$x^2 = 1 \quad x = \pm 1$

$\Rightarrow$ 2 points $(-1, 2)$ and $(1, 2)$

(upper function: $g_2(x) = 1 + x^2$)
(lower function: $g_1(x) = 2x^2$)

\[ \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x+2y) \, dy \, dx = \int_{-1}^{1} (xy + y^2) \bigg|_{2x^2}^{1+x^2} \, dx \]

\[ = \int_{-1}^{1} (x \left(1 + x^2\right) + (1 + x^2)^2 - 2x^3 - 4x^4) \, dx \]

\[ = \int_{-1}^{1} (x + x^3 + 1 + 2x^2 + x^4 - 2x^3 - 4x^4) \, dx \]

\[ = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1) \, dx \]

\[ = \left( -\frac{3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x \right) \bigg|_{-1}^{1} \]

$\Rightarrow$ I'll let you calculate this then.