Simple, Compound and Continuously compounded interests.

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An investor wants to give the bank a fixed (constant) amount of money $P_0$ and the bank offers some interest for the money. There are few schemes in which the bank can pay these interests. Assume we are working with a fixed unit of time (example, a year).

§ 1. Simple interest

In this scheme, the bank pays the investor $rP_0$ at the end of the period of time (example a year). So, simple interest with rate $r \in (0, 1]$ (so that $100r\%$ is the interest) is just $rP_0$ and the balance at the end of the first period is $P_0 + rP_0 = P_0(1 + r)$. A modification of this is when the bank decides to pay the investor in $n$ subperiods of time equally divided (example, monthly, quarterely, half-yearly), then the bank pays $\frac{r}{n}P_0$ after each subperiod of time, so that the investor has balance:

<table>
<thead>
<tr>
<th>Subperiods:</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>$\ldots$</th>
<th>$n$ (one complete period)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balance:</td>
<td>$P_0$</td>
<td>$P_0\left(1 + \frac{r}{n}\right)$</td>
<td>$P_0\left(1 + \frac{2r}{n}\right)$</td>
<td>$\ldots$</td>
<td>$P_0(1 + r)$</td>
</tr>
</tbody>
</table>

§ 2. Compounded interest.

In this scheme, the bank will pay simple interest in each subperiod of time but with the condition that after each subperiod has elapsed, the bank refreshes the initial investment to the balance at the end of the previous subperiod (so that the bank gives you interest based on the money you currently have and not the money you had several subperiods ago). By definition, the bank gives the investor $\frac{r}{n}P_0$ at the end of the first subperiod and so, when $\frac{1}{n}$ of the total period has passed, the investor has $P_{\frac{1}{n}} = P_0\left(1 + \frac{r}{n}\right)$, this will become the new investment for the subperiod 2. Repeating the argument, at the end of subperiod 2, the bank gives $\frac{r}{n}P_{\frac{1}{n}}$ and the new balance is $P_{\frac{2}{n}} = P_{\frac{1}{n}} + \frac{r}{n}P_{\frac{1}{n}} = P_{\frac{n}{n}}\left(1 + \frac{r}{n}\right) = P_0\left(1 + \frac{r}{n}\right)^2$. In general, when $k$ of the subperiods have just passed, so that $\frac{k}{n}$ of the total time have elapsed, the investor has balance:

$$P_{\frac{k}{n}} = P_0\left(1 + \frac{r}{n}\right)^k \quad \text{for } k = 1, 2, 3, \ldots$$

Notice that when $k = n$, that is, when a whole period has passed, the investor’s balance is $P_1 = P_0\left(1 + \frac{r}{n}\right)^n$ which is known as compounded interest over a fixed period of time with $n$ subdivisions of the time.
§ 3. Continuously compounded interest.

We can manipulate algebraically the previous formula inside the box to get

\[ P_k = P_0 \left( 1 + \frac{r}{n} \right)^{\frac{k}{n}} = P_0 \left[ \left( 1 + \frac{r}{n} \right)^n \right]^{\frac{k}{n}} \quad \text{for } k = 1, 2, \ldots, n. \]

Assume that \( k \) and \( n \) move in such a way that \( n \rightarrow \infty \) (that is, \( n \) becomes larger and larger without bound) and as \( n \rightarrow \infty \), the quotient \( \frac{k}{n} \) has a limit \( t \in [0, 1] \) (for example, if we always look at the end of the subperiod in the middle, \( k \) will be either \( \frac{n}{2} \) when \( n \) is even or \( \frac{n+1}{2} \) when \( n \) is odd, in both of these cases \( \frac{k}{n} \) has limit \( \frac{1}{2} \); one can look more general \( k \) to reflect any instant during the whole period). Then, the outside most exponent will tend or converge to \( t \). Jacob Bernoulli in 1683 showed that the expression inside brackets converges towards \( e^r \), when \( n \rightarrow \infty \) and where \( r \mapsto e^r \) is the standard exponential function. Therefore, limit laws for exponents allows deducing that as \( n \) tends to infinity, the right hand side above has a limit and that this limit equals \( P_0 [e^r]^t = P_0 e^{rt} \). Therefore, it is sensible to say that continuously compounded interest should be defined as \( P_t = P_0 e^{rt} \). This is what is done and so, the definition of continuously compounded interest at \( t \) units of time (whatever \( t \geq 0 \) maybe, whole number or not) starting at time \( t = 0 \) with \( P_0 \) interest and interest rate \( r \in (0, 1] \) (or 100% interest) is

\[ P_t = P_0 e^{rt}, \quad \text{for } t \geq 0. \]

§ 4. Practice exercises.

Exercise (4.1) If $100 is invested during a year using simple interest with monthly payments at a rate of 3% interest, how much money will the investor have after five months?

Exercise (4.2) If a $100 is invested at 8% interest compounded quarterly, how much money will the investor have after one year? How about two years?

Exercise (4.3) A bank pays investors 10% of interests at the end of the year and then investor can reinvest (at the same rate) or withdraw part or total of their money. Assume you invest in this model with a start of \( P_0 = $100 \) dollars.

1. How many years will the money should be invested without withdraws to double the original investing?

2. At the end of the first year, you decide to withdraw $70 because of an emergency. How many years will you need to invest your money without withdraw to have again at least the original amount \( P_0 \).

Exercise (4.4) There is economical crisis and inflation is sky-high at a rate of 12% annually. How many years will need to elapse from now until prices have reached twice of today’s price. Assume that the inflation is continuously happening and answer the same question. Do you find this result surprising?

Exercise (4.5) A healthy economy has inflation under 3%. If the inflation were exactly 3%, how much time does it need for prices to double? If it were, 2%, how much time? Answer the same questions for continuously compounded interest.