Exercise (1) In economics, “utility” is a term used to represent satisfaction or wellness over the consumption of good. Suppose that your utility is given by $U = x^3 y^2 z^4$, where $x$ is the number of hours you sleep, $y$ is the amount of leisure time and $z$ is the time spent studying. Imagine the model is set-up such that all your day time is spent only on these three activities. Suppose additionally that you value sleeping three times as much as leisure activities, and so, this is reflected in the amount of time spent on these two things. How should you spent your time to maximise utility? You must justify that the proposed trio of times maximises utility; any unjustified answer shall be penalised heavily in marks.

Hint. One day has 24 hours and the conditions of the problem imply $x + y + z = 24$. Similarly, the condition “sleeping three times as much as leisure” implies $x = 3y$. Express $z = f(y)$, that is, express $z$ in terms of $y$ only; then substitute $x = 3y$ and $z = f(y)$ into $U$, to get $U = (3y)^3 y^2 f(y)^4$, which is the function you have to maximise on some domain. To find the domain of $U$ as a function of $y$ remark that $0 \leq x + y \leq 24$ and $x = 3y$.

Proposed solution. As in the hint, we can write $z = 24 - 4y$, so that $U = \sqrt[3]{3} y^2 (24 - 4y)^4$ and its domain is $0 \leq 4y \leq 24$ or $0 \leq y \leq 6$. Suppose that $0 < y < 6$ to find candidates inside $(0, 6)$. Logarithmic differentiation of $U$ yields $(\ln U)' = \frac{U'}{U}$ or, equivalently, $U' = U (\ln U)' = U \left( \frac{2}{3} \times \frac{1}{y} + \frac{3}{4} \times -\frac{4}{24 - y} \right)$. This will be zero if and only if (since we assumed $y \neq 0$) $\frac{2}{3y} = \frac{4}{3(24 - 4y)}$ or $2(24 - 4y) = 4y$ or $48 = 12y$ or $y^* = 4$. Observe that $U(0) = U(6) = 0$ and $U(4) > 0$, so $y^*$ is global maximiser by the fundamental extreme value theorem. You should therefore spent $y^* = 4$ hours in leisure time, $x^* = 3y^* = 12$ hours sleeping and $z^* = 24 - 4y^* = 8$ hours studying. ♠

Exercise (2) Supertankers carrying oil to a refinery will discharge their oil to a deep water pumping station one kilometre off-shore. The refinery is located three kilometres along the shore, which is straight. Suppose that constructing underwater pipelines costs five times as much per kilometre as one on land. How much pipeline should be constructed in land and how much underwater in such a way that the pipeline connects the refinery with the pumping station and the cost is minimised? You must justify that the proposed lengths of pipelines minimises cost; any unjustified answer shall be penalised heavily in marks.

Hint. The shore can be thought as a straight line and the refinery on a line perpendicular to the shore, say the $x$ and $y$ axis; for simplicity, use one unit to represent kilometres. The refinery is then located at $(3, 0)$ and the water pumping station at $(0, 1)$ (draw a diagram to convince yourself of this). You want to select a point $(s, 0)$ in the shore such that the segment joining the two points $(0, 1)$ and $(s, 0)$ is the underwater pipeline and the segment joining the two points $(s, 0)$ and $(3, 0)$ is the land pipeline. Construct a function $H(s)$ that represent the cost of the construction if the point $(s, 0)$ is selected and observe that $s$ can be selected anywhere between $0$ and $3$, inclusive.

Proposed solution. As in the hint, suppose that $H = H(s)$ is the cost of building the pipeline selecting the point $(s, 0)$. The domain of $H$ is $[0, 3]$. Then, $H = 5\sqrt{1 + s^2} + 3 - s$, and therefore $H' = \frac{5s}{\sqrt{1 + s^2}} - 1$.

Observe that $H' = 0$ is the same as $5s = \sqrt{1 + s^2}$ or $25s^2 = 1 + s^2$, so that $s^* = \frac{1}{\sqrt{24}}$ (since $s > 0$). Now, $H(0) = 8$, $H(3) = 5\sqrt{10}$ and $H(s^*) = 5\sqrt{1 + \frac{1}{24} + 3 - \frac{1}{\sqrt{24}}} = 5\sqrt{\frac{25}{24} + 3} - \frac{1}{\sqrt{24}} = \frac{25}{\sqrt{24}} + 3 = \frac{24}{\sqrt{24}} + 3 = \sqrt{24} + 3 < \sqrt{25} + 3 = 8 = H(0)$. Henceforth, $H(s^*)$ is a minimum and $s^*$ is a minimiser by the fundamental extreme value theorem. The construction should include $\sqrt{2524}$ kilometres of underwater pipeline and $3 - \frac{1}{\sqrt{24}}$ kilometres of land pipeline. ♠
Exercise (3) A company can sell \( q = D(p) = \sqrt{100 - p} \) tons of produce when the price is set at \( p \) dollars per ton. If each ton costs $10 to produce, what price will maximise the profit? **You must justify that the proposed price maximises profit; any unjustified answer shall be penalised heavily in marks.**

**Hint.** Recall the definition of profit as something relating revenue and cost; revenue is the product of two variables and one of these is expressed in terms of the other one; cost in this case is just the amount paid for all produced tons. Observe that price cannot be negative and that quantity cannot be negative either.  

**Proposed solution.** Remark that profit is \( P = R - C \), where \( R \) is revenue and \( C \) is cost. By definition \( R = pq = p\sqrt{100 - p} \) and \( C = 10q = 10\sqrt{100 - p} \). Therefore, \( P = (p - 10)\sqrt{100 - p} \). Observe that \( p \geq 0 \) and that \( q \geq 0 \), that is \( 100 - p \geq 0 \) or \( p \leq 100 \). So, we want to maximise \( P = P(p) \) with \( 0 \leq p \leq 100 \).

Now, \( P' = \sqrt{100 - p} + (p - 10)\frac{1}{2\sqrt{100 - p}}(-1) \) and \( P' = 0 \) if and only if \( \sqrt{100 - p} = \frac{p - 10}{2\sqrt{100 - p}} \) or, equivalently, \( 200 - 2p = p - 10 \), that is \( p^* = \frac{190}{3} \). Observe that \( P(0) = 0 \), \( P(p^*) > 0 \) and \( P(100) = 0 \), so \( p^* \) is a global maximiser by the fundamental extreme value theorem. ♣

Exercise (4) A printer is to produce 100,000 identical posters by using printing blocks that each print 100 posters per hour (so that, for example, if the printer uses only 2 blocks it would take 500 hours to do the job). Each block costs $2.00 to make, and no matter how many blocks the printer uses at the same time, the overhead expenses are $15.00 per hour. How many blocks should be used to produce the posters at minimum cost? **You must justify that the proposed number of blocks minimises cost; any unjustified answer shall be penalised heavily in marks.**

**Hint.** Start by defining \( n \) the number of blocks made and \( t \) the time used in number of hours, for instance, \( t = 1 \) means one hour. The condition “no matter how many blocks the printer uses at the same time, the overhead expenses are $15.00 per hour” means that part of the cost is 15\( t \). The condition “Each block costs $2.00 to make” means that the cost of the blocks is 2\( n \). So, the total cost is \( C = 2n + 15t \). You need now to find a relation between \( n \) and \( t \) and this is given in “A printer is to produce 100,000 identical posters by using printing blocks that each print 100 posters per hour” which can be translated into \( 100t \) is the number of posters in one hour produced by each block, and we want to produce a 100,000, so that 100\( nt = 100,000 \).

**Proposed solution.** As in the hint, \( C = 2n + 15t \) and \( nt = 1000 \), so \( n = \frac{1000}{t} \). Therefore, \( C = \frac{2000}{t} + 15t \), with \( t > 0 \). Notice that \( C' = -\frac{2000}{t^2} + 15 \) and \( C'' = \frac{4000}{t^3} > 0 \) for \( t > 0 \). Now, \( C' = 0 \) if and only if \( \frac{2000}{t^2} = 15 \) or \( t^2 = \frac{2000}{15} \) or \( t^* = \sqrt{\frac{2000}{15}} \) (since \( t > 0 \)). Since \( C'' > 0 \) everywhere on the domain \( t > 0 \), any point where the first derivative is zero is a global minimum by one of the two parts of the second order criteria. Therefore, \( n^* = \frac{1000}{t^*} \) and \( t^* = \sqrt{\frac{2000}{15}} \) is an optimisation pair. ♣