REMARK (1) The following theorem is fundamental for optimisation, but its proof (as the proof of the intermediate value theorem), is outside the scope of the course.

**Theorem:** for any continuous function \( f : [a, b] \rightarrow \mathbb{R} \) there exists a point \( x^* \) such that \( f(x^*) \geq f(x) \) for all \( x \in [a, b] \).

In sight of the inequality \( f(x^*) \geq f(x) \), which means that \( f \) attains its largest \( y \)-value at \( x^* \), we will call \( f(x^*) \) the **maximum of** \( f \) and we will call \( x^* \) the **maximiser**. Notice that maximisers are hardly unique, as the parabola \( x^2 \) shows in \([-1, 1] \), both \( x_1^* = -1 \) and \( x_2^* = 1 \) are maximisers. The definition of **minimisers** and **minimum** are similar and its existence is guaranteed by an analogue theorem to the previous one. Often enough, we will talk of **extremum** and **extrema**, these are synonyms (the same meaning as) optimiser and optimisers, respectively. When we deal with both maximum and minimum, we will talk about **optimum values** and the corresponding maximiser and minimiser the **optimisers**. Note: the plural of minimum is minima, of maximum, maxima and of optimum, optima. The previous theorem can then be summarised in the following.

A continuous function on an interval of the form \( [a, b] \) have both maximum and minimum.

The question that follows is: how to find such optimisers? In general, there is no hope. However, when the function is also differentiable on \((a, b)\), we can calculate the derivative. Assume for the time being that \( x^* \) is a maximiser of \( f \) in \((a, b)\). Then, in the quotient \( \frac{f(x^* + h) - f(x^*)}{h} \), the numerator \( f(x^* + h) - f(x^*) \) is always negative, so the quotient has negative sign for \( h > 0 \) and positive sign for \( h < 0 \), so if the limit exists, it can only be zero (zero is the only number that is arbitrarily close to negative and positive numbers, any negative number is surrounded by negative numbers and, similarly, any positive number is surrounded by positive numbers). So, it must be the case that \( f'(x^*) = 0 \). Notice that at \( a \) and at \( b \) the definition of derivative does not make sense.

A continuous function \( f : [a, b] \rightarrow \mathbb{R} \) that is also differentiable on \((a, b)\) will either have optimisers \( x^* \in (a, b) \) satisfying \( f'(x^*) = 0 \) or else, the optimisers are \( a \) or \( b \) (or both).

As an example, consider the parabola again \( x^2 \) on \([-1, 1]\). The derivative of the parabola is \( 2x \) and so, it is zero if and only if \( x = 0 \). This does not imply that \( x \) is an optimiser! However, if the parabola has an optimiser inside \((-1, 1)\) it can only be \( x = 0 \). The other two candidates are \(-1 \) and \(+1 \). Observe that amongst these three numbers \(-1, 0, 1\) there must be at least one of each: minimiser & maximiser. To find what is what, evaluate the function in each of them. Then, \( f(-1) = 1 \), \( f(0) = 0 \) and \( f(1) = 1 \). Therefore, 0 is a minimiser and \(-1, 1 \) are maximisers. In the case, 0 in the minimum of \( f \) and 1 the maximum.

In general, one can find all the optimiser candidates \( x_1^*, x_2^*, \ldots, x_p^* \) (say, there are \( p \) of them) inside the interval \((a, b)\) and the two extreme points of this interval \( a \) and \( b \). This creates a list of numbers \( a, x_1^*, \ldots, x_p^*, b \) and we can calculate the value of the function at each one of them \( f(a), f(x_1^*), \ldots, f(x_p^*), f(b) \). Since the optimisers exist, those points from the list for which the \( y \)-value is minimum are minimisers and those for which the \( y \)-value is maximum are maximisers.

Consider, for example, the function \( x \mapsto x^3 + 6x^2 - 15x - 4 \) on the interval \([0, 3]\). Then, the derivative of this function is given by \( 3x^2 + 12x - 15 \) and this is zero if and only if \( x^2 + 4x - 5 = 0 \) and we can factor this as \((x - 1)(x + 5) = 0 \) so we obtain the two candidates \( x_1^* = 1 \) and \( x_2^* = -5 \). Since \(-5 \) is not in the interval \([0, 3]\), we must discard it. So, we are left with one candidate \( x^* = 1 \). Then, we have a list with three points \( 0, 1, 3 \). Observe that the function have corresponding \( y \)-values \(-4, -12, 32 \). So, \( x^* = 1 \) is a minimiser with minimum value \(-12 \) and 3 is a maximiser with maximum value 32.

**Exercise (2)** Find all optima and optimisers for the following functions on the given intervals.
a) \( x^2 + 3x + 1 \) on \([2, 3]\).

b) \( x^3 - x \) on \([-1, 2]\).

c) \( x^3 - x^2 - 8x + 3 \) on \([-2, 2]\).

d) \( \frac{1}{1 + x^2} \) on \([-3, 3]\).

e) \( e^{-(x-2)^2} \) on \([0, 5]\).

Remark (3) Consider now a “wordy” example. Imagine that it is required to build a fence enclosing a rectangle of given area 200 square metres, with one end facing a highway. The shape of the rectangle doesn’t matter as long as it has the determined area. The end facing the highway needs a decorative fence costing $18 per metre, while the other boundaries can have a cheaper fence costing $5 per metre. What should the lengths of the sides of the rectangle be in order to minimise the cost of the fence? To proceed with such a problem one needs to extract the information provided. We have a rectangle with two sides a long one, call it \( l \), and a wide one, call it \( w \) (we are just using common language to name the length of the sides, it can perfectly be the case that after the calculations are performed, \( w > l \), that is, the long side is smaller than the wide one, this could be the case and it doesn’t matter. If you prefer, just call the length of the sides, \( a \) and \( b \) or \( x \) and \( y \)). Assume that one of the long sides is facing the highway. Then, the cost is given by

\[ C = 18l + 5l + 5w + 5w = 23l + 10w. \]

Since we are dealing with a rectangle, elementary geometry shows \( wl = 200 \) or, for example, \( w = \frac{200}{l} \). This in turn implies \( C = 23l + \frac{2000}{l} \), which is the function that we want to optimise (being optimum in this case means being a minimum). The domain of the function \( C \) are all those values of \( l > 0 \). We are not dealing in a closed interval \([a, b]\) but on the open interval \((0, \infty)\), I’ll justify that our candidate is an optimiser at the end. Observe that as before, if \( l^* \in (0, \infty) \) is an optimiser, it must be the case that \( C'(l^*) = 0 \), so, for the time being, we want to find \( l^* \) such that \( C'(l^*) = 0 \). We know that \( C' = 23 - \frac{2000}{l^2} \) and so, \( C' = 0 \) if and only if \( 23 = \frac{2000}{l^2} \) or, equivalently, \( l^2 = \frac{2000}{23} \). Since \( l > 0 \), we must have \( l^* = \sqrt{\frac{2000}{23}} \). I will now justify that \( l^* \) is a minimiser, I will heavily use the intermediate value theorem. Observe that \( C' = 23 - \frac{2000}{l^2} \) is a continuous function of \( l \) (on its domain \( l > 0 \)) and it only has one zero, when \( l = l^* \). Therefore, \( C'(l) \) will be positive for all \( l > l^* \) or else, \( C'(l) \) will be negative for all \( l > l^* \) (if it were both positive and negative for two values larger that \( l^* \), the intermediate value theorem would imply that \( C'(l) = 0 \) for some value \( l > l^* \) and this would violate the fact proved about \( l^* \) being the only root of \( C' \)). However, it should be clear now that \( C'(l) \) is positive for all \( l > l^* \) because we could take, say, \( l \) to be one million and then \( \frac{2000}{l^2} \) will be almost zero making \( C'(l) \) to be about 23, which is a positive number. From here, since \( C'(l) > 0 \) for all \( l > l^* \), \( C \) is increasing on the interval \((l^*, \infty)\) and, therefore, \( C(l) \) is larger than \( C(l^*) \) for \( l > l^* \). In a similar fashion, the same arguments provided state that \( C'(l) \) is always negative for all \( l < l^* \) or is always positive for all \( l < l^* \). But since \( C'(0.1) = 23 - \text{very large number} < 0 \), \( C'(l) \) is negative on \((0, l^*)\). This shows \( C(l) \) is decreasing on \((0, l^*)\) and, therefore, the values \( C(l) \) on the left of \( l^* \) are larger than the value \( C(l^*) \). Insofar as the example goes, we have established that \( C(l) \) is larger both to the right and to the left of \( l^* \), which is the same as saying that \( l^* \) is a minimiser of \( C \). Notice then that the answer to the question of the problem is \( l^* = \sqrt{\frac{2000}{3}} \) and \( w^* = \frac{200}{l^*} = \frac{200}{\sqrt{\frac{2000}{3}}} \).

The last example can be carried out word by word in a totally general context. If a function \( f \) is defined on an open interval \((a, b)\), possibly of the forms \((a, \infty)\), \((-\infty, b)\) or \((-\infty, \infty)\) and \( x^* \) is a point in this interval
satisfying that \( f'(x) < 0 \) for \( x < x^* \) and \( f'(x) > 0 \) for \( x > x^* \) then \( f \) is decreasing on \((a, x^*)\) and increasing on \((x^*, b)\) and so, \( x^* \) is a minimiser. A similar result holds true for maximisers.

**First derivative test:** suppose that \( f \) is a differentiable function on an open interval \((a, b)\) and that \( x^* \in (a, b) \) satisfies \( f'(x) > 0 \) (respectively \( f'(x) < 0 \)) for all \( x < x^* \) (or all \( x > x^* \)) if and only if \( f'(x^*) > 0 \) (respectively, \( f'(x^*) < 0 \)) for one value \( x < x^* \) (respectively, or some value \( x > x^* \)).

And the argumentation given above using the intermediate value theorem can allows us to reduce to check the sign of the derivative in just one point on each side of the optimiser as opposed on the entire interval.

If \( f \) is a differentiable function on \((a, b)\) such that \( f' \) is a continuous function on \((a, b)\) and \( x^* \) is a zero (a root) of \( f' \) then \( f'(x) > 0 \) (respectively, \( f'(x) < 0 \)) for all \( x < x^* \) (or all \( x > x^* \)) if and only if \( f'(x) > 0 \) (respectively, \( f'(x) < 0 \)) for one value \( x < x^* \) (respectively, or some value \( x > x^* \)).

**Exercise (4)** An open box (no top) with square base is to have a volume of 81 cubic centimetre. If the material for the base costs $9 per square centimetre, and the material for the sides costs $3 per square centimetre, what should be the dimensions of the box in order that the cost be minimized?

**Exercise (5)** Plans for a new rectangular building require a floor area of 12,300 square metres. The walls are 7 metres high. Three walls are made of brick and the fourth wall is made of glass. Glass costs 2.5 times as much as brick per square metre. What should the dimensions of the building be so that the cost for the walls is minimum?

**Exercise (6)** A farmer has 200 metres of fencing to enclose a rectangular pasture adjacent to a long stone wall. What dimensions give the maximum area of the pasture?

**Exercise (7)** A manufacturer has determined that for a certain product the (average) cost function is \( C = q^2 + 30q - 400 + \frac{200}{q} \) and the demand equation is \( p = 2q^2 - 15q + 200 + \frac{300}{q} \), both these relations being valid for values of \( q \) in the closed interval \([1, 21]\). At what level within this interval should production be fixed to maximize profit? What is the maximum profit obtainable? At what price per unit is this profit obtained?

**Exercise (8)** A truck is to be driven 130 kilometres at a constant speed of \( x \) kilometres per hour. Speed laws require that \( 50 \leq x \leq 100 \). Assume that gasoline costs 116 cents per litre and that it is consumed at the rate of \( 2 + \frac{x^2}{270} \) litres per hour. If the driver is paid 15 dollars per hour, find the most economical speed and the total cost for the trip.