Maths 104:103 2017W.
EIGHT AND LAST WRITTEN ASSIGNMENT.
For Friday, December 1st, 2017 at the beginning of lecture.

Note. No assignment shall be received later.

Comment for the student. The purpose of the assignment is to clarify what are we doing in this last week of the course. This assignment focuses in finite expansions or finite developments. In particular, we are considering development by polynomials of zeroth, first and second degree.

Remark (8.1) So far we have seen two types of “finite developments”, “finite expansions”, “finite series” or “approximations” (all these are synonyms that refer to the following lines):

\[
(8.2) \quad f(a + h) = f(a) + R_0(a; h)
\]

Constant approximation  Zeroth order remainder

and

\[
(8.3) \quad f(a + h) = f(a) + hf'(a) + R_1(a; h)
\]

Linear approximation  First order remainder

The point \( a \) is called “centre” of expansion and \( h \) the “increment.”

These expansions are useless unless we know how big are these “error” terms or “remainders” \( R_0 \) and \( R_1 \). The best bounds for the largest possible value of the remainder (the “worst-case” error)

\[
(8.4) \quad |R_0(a; h)| \leq M_1|h|, \quad |R_1(a; h)| \leq \frac{M_2}{2} |h|
\]

where \( M_1 \) and \( M_2 \) are positive numbers that bound the first and second derivative on the interval \([a, a+h]\) in case \( h > 0 \) and on \([a+h, a]\) in case \( h < 0 \). So, \( M_1 \) and \( M_2 \) must satisfy

\[
(8.5) \quad |f'(x)| \leq M_1, \quad |f''(x)| \leq M_2, \quad \text{for all } x \text{ between } a \text{ and } a+h.
\]

In a similar fashion, we can work with a “second order” or “quadratic” approximation:

\[
(8.6) \quad f(a + h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a) + R_2(a; h)
\]

Quadratic approximation  Second order remainder
And, as before, the best bound for the worst-case error is given by

\[(8.7) \quad |R_2(a; h)| \leq \frac{M_2}{6} |h|^3, \quad |f'''(x)| \leq M_3, \quad \text{for all } x \text{ between } a \text{ and } a + h.\]

It must be emphasised that

A function that is \( p \) times differentiable (or more differentiable) has a unique approximation of \( p \)-th order.

ILLUSTRATIVE EXAMPLE (8.8) Find the constant, linear and quadratic approximation of \( e^x \) near \( a = 0 \) and of \( \ln(x) \) near \( a = 1 \). Approximate \( e^{0.1} \) and \( \ln(0.75) \) using each of these three approximations. Find bounds as good as possible and as explicit as possible for the error term for each one of them.

Proposed solution. Consider the functions \( f(x) = e^x \) and \( g(x) = \ln(x) \). Then, \( f'(x) = f''(x) = f'''(x) = e^x \), \( g'(x) = \frac{1}{x} \), \( g''(x) = -\frac{1}{x^2} \) and \( g'''(x) = \frac{2}{x^3} \).

**Constant approximation.** By looking at (8.2), we know that the constant approximations are

\[ e^{0.1} \approx e^0 = 1, \quad \ln(0.75) \approx \ln(1) = 0. \]

Since we are dealing with two different functions, let us denote the error terms of \( f \) with \( R \) and the error terms of \( g \) by \( H \). Then, the error terms are bounded by

\[ R_0(0; 0.1) \leq M_1 |0.1|, \quad H_0(1; -0.25) \leq N_1 |0.25|, \]

where \( M_1 \) bounds the derivative of \( f \) on \([0, 0.1]\) and \( N_1 \) bounds the derivative of \( g \) on \([0.75, 1]\). So, the task is to find \( M_1 \) such that \( e^x \leq M_1 \) for \( 0 \leq x \leq 0.1 \), since we want an explicit bound we can say the following \( e^x < e^{0.3} < 3^{0.5} = \sqrt{3} < 1.8 \) (because \( 1.8^2 = 3.24 \)), so we can take \( M_1 = 1.8 \). Similarly, we want \( N_1 \) such that \( \frac{1}{x} < N_1 \) for all \( x \in [0.75, 1] \). The fraction is largest when \( x \) is the smallest, so take \( x = 0.75 \) and observe that \( \frac{1}{0.75} = \frac{4}{3} = 1.3333... < 1.5 \). We can take \( N_1 = 1.5 \). Therefore, \( |R_1(0; 0.1)| \leq (1.8)(0.1) = 0.18 \) and \( |H_1(1; -0.25)| \leq (1.5)(0.25) = 0.375 \). This means that, in fact,

\[ 1 - 0.18 = 0.82 \leq e^{0.1} \leq 1.18 = 1 + 0.18, \quad 0 - 0.375 = 0.375 \leq \ln(0.75) \leq 0.375 = 0 + 0.375. \]

**Linear approximation.** Since \( f'(0) = e^0 = 1 \) and \( g'(1) = 1 \), we get

\[ e^{0.1} \approx e^0 + 0.1 e^0 = 1.1, \quad \ln(0.75) \approx \ln(1) + (-0.25) \frac{1}{1} = -0.25. \]

Then, the error terms are bounded by

\[ |R_1(0; 0.1)| \leq \frac{M_2}{2} |0.1|^2, \quad |H_1(1; -0.25)| \leq \frac{N_2}{2} |0.25|^2, \]

Since the first and second derivative of \( f \) are the same, we can take \( M_2 = 1.8 \) as before. We want \( N_2 \) such that \( \frac{1}{x^2} \leq N_2 \) for all \( x \in [0.75, 1] \), so again we take \( x = 0.75 \) (which gives the largest fraction) and we can take \( \frac{1}{0.75^2} = \left(\frac{4}{3}\right)^2 < 1.5^2 = 2.25 = N_2 \). Therefore,

\[ |R_1(0; 0.1)| \leq \left(\frac{1.8}{2}\right)(0.1)^2 = 0.009, \quad |H_1(0; 0.1)| \leq \left(\frac{2.25}{2}\right)(0.25)^2 = 0.0703125 < 0.071. \]
This means that
\[ 1.1 - 0.009 = 1.091 \leq e^{0.1} \leq 1.109 = 1 + 0.009, \]
and that
\[ -0.25 - 0.071 = -0.321 \leq \ln(0.75) \leq -0.179 = -0.25 + 0.071. \]

**Quadratic approximation.** Since \( f''(0) = e^0 = 1 \) and \( g''(1) = -1 \), we have
\[
e^{0.1} \approx 1 + 0.1 + \frac{0.1^2}{2} = 1.105, \quad \ln(0.75) \approx 0 + (-0.25) + \frac{(-0.25)^2}{2} = -0.25 - 0.03 + 0.0625 = 0.091 = 1.091 \leq e^{0.1} \leq 1.109 = 1 + 0.009,
\]
We can take \( M_3 = 1.8 \) (because the third derivative of \( f \) is just \( f \)) and since \( \frac{2}{0.75^3} = \frac{(2)(4^3)}{3^3} = \frac{128}{27} < 130 = 4.333... < 4.5 \), we can take \( N_3 = 4.5 \). Then,
\[
|R_2(0; 0.1)| \leq \left( \frac{1.8}{6} \right) (0.1)^3 = 0.0003, \quad |H_2(1; -0.25)| \leq \left( \frac{4.5}{6} \right) (0.25)^3 = 0.01171875 < 0.012.
\]
This means,
\[ 1.105 - 0.0003 = 1.1047 \leq e^{0.1} \leq 1.1053 = 1.105 + 0.0003
\]
and that
\[ -0.28125 - 0.012 = -0.29325 \leq \ln(0.75) \leq -0.26925 = -0.28125 + 0.012.
\]

**Actual values from computer.**
\[ e^{0.1} = 1.10517091808, \quad \ln(0.75) = -0.28768207245. \]

This concludes the example.

**Illustrative example (8.9)** Suppose that \( f(x) \approx 2x^2 - 3x + 1 \) around \( a = 2 \). Find \( f(2), f'(2) \) and \( f''(2) \).

**Proposed solution 1.** Make the change of variable \( x = 2 + h \). Then \( f(2 + h) \approx 2(2 + h)^2 - 3(2 + h) + 1 = 2(4 + 4h + h^2) - 6 - 3h + 1 = 2h^2 + 5h + 3 = \frac{f''(2)}{2}h^2 + f'(2)h + f(2) \). Uniqueness of expansion then shows at once \( \frac{f''(2)}{2} = 2 \) or \( f''(2) = 4 \), \( f'(2) = 5 \) and \( f(2) = 3 \).

**Proposed solution 2.** Bearing in mind \( x = 2 + h \) and that \( x \approx 2 \) is the same as \( h \approx 0 \), we get that \( f(2) = 2(2)^2 - 3(2) + 1 = 8 - 6 + 1 = 3, f'(2) = 4(2) - 3 = 5 \) and \( f''(2) = 4 \).

**Exercise (8.10)** Find the constant, linear and quadratic approximation of the function \( x \mapsto \frac{1}{1+x} \) around \( 0 \) and around \( 1 \). Using these approximations, estimate \( \frac{1}{1 + \frac{1}{2}} = 0.75 \). Find the increment for each centre of expansion, find explicit error bounds and give intervals where the approximation lies. Compare your approximations with the actual value of 0.75. The grading scheme of this problem is entirely up to the TA.