Note. No assignment shall be receive in the afternoon or later.

Comment for the student. The purpose of the assignment is for you to practice how to write mathematics and no so much on the content of the course. Write the assignment as if you were explaining it for one of your schoolmates. Correctness of ideas and explanations are going to receive more marks than the actual result itself. Use lay language (or common use language) and do not make an essay out of this. This exercise should fit perfectly in one page. An assignment where no explanation is provided of what is going on shall receive a mark of zero, even if you draw, wrote formulae or equations that make sense.

Remark (3.1) A function $E$ defined on an interval around a point $0$ is said to be “vanishing faster than linear” (or that $E$ is “negligible in comparison to linear”) if $\lim_{x \to 0} \frac{E(x)}{x} = 0$. This intuitively says that $E$ is a function that eventually around zero will be less than whatever fraction of $x$ maybe be (it will be less than $\frac{1}{x}$ eventually as $x$ approaches zero, or than $\frac{x}{10^{10}}$ eventually as $x$ approaches zero, etc) and observe that $x$ is also tending to zero, so $E$ is approaching zero really really fast, in other words, $E$ is “infinitesimal.”

Exercise (3.2) Consider a function $f$ defined on some interval $(a, b)$ into the real numbers $\mathbb{R}$. Suppose $f$ is differentiable at the point $c \in (a, b)$. If $L$ is a line $L(x) = mx + b$ satisfying the following “first order” approximation of $f$ at $c$:

$$f(c + x) = L(x) + \text{error}(x)$$

where $x \mapsto \text{error}(x)$ is a function defined for $x$ around zero that vanishes faster than linear, then $m = f'(c)$ and $b = f(c)$. That is, show that $L$ is the tangent line to the function $f$ at $(c, f(c))$, $L(x) = f'(c)x + f(c)$. Therefore, the exists a line $L$ and only one line that is an approximation of $f(c + x)$ (as $x \to 0$) that makes the difference $f(c + x) - L(x)$ negligible in comparison to linear.

Remark (3.3) In sight of the previous exercise, we will call the function $x \mapsto f'(c)x + f(c)$ the best “linear approximation” to $f$ at $c + x$. For instance, if we wanted to approximate $\sqrt{101}$ we could take $f(x) = \sqrt{x}$, $c = 100$ and $x = 1$, then $\sqrt{101} = \frac{1}{2\sqrt{100}}1 + \sqrt{100} = \frac{1}{20} + 10 = 10.05$. (A calculator gives $10.0498756$, so our approximation is very good for being so simple).

Exercise (3.4) Prove the following inequality $e^x \geq 1 + x$ for all $-\infty < x < \infty$. Below there are two pictures of both functions (far and close) and it shows that this inequality is plausible, however, a picture is not a proof.

Hint: using the fact that $e^x$ is never negative, reduce to the case of proving the inequality for $-1 \leq x < \infty$. Then, divide the interval into two parts $-1 \leq x \leq 0$ and $0 \leq x < \infty$. Apply the Mean Value Theorem on the interval $[-1, 0]$ and on $[0, N]$ for each positive integer $N$. 

Hint: using the fact that $e^x$ is never negative, reduce to the case of proving the inequality for $-1 \leq x < \infty$. Then, divide the interval into two parts $-1 \leq x \leq 0$ and $0 \leq x < \infty$. Apply the Mean Value Theorem on the interval $[-1, 0]$ and on $[0, N]$ for each positive integer $N$. 

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