PROOF OF A DYNAMICAL BOGOMOLOV CONJECTURE
FOR LINES UNDER POLYNOMIAL ACTIONS

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Abstract. We prove a dynamical version of the Bogomolov conjecture
in the special case of lines in $\mathbb{A}^m$ under the action of a map $(f_1, \ldots, f_m)$
where each $f_i$ is a polynomial in $\mathbb{Q}[X]$ of the same degree.

1. Introduction

In 1998, Ullmo [Ull98] and Zhang [Zha98] proved the following conjecture
of Bogomolov [Bog91].

Theorem 1.1. Let $A$ be an abelian variety defined over a number field with
Néron-Tate height $\hat{h}_{nt}$ and let $W$ be an irreducible subvariety of $A$ that is
not a torsion translate of an abelian subvariety of $A$. Then there exists an
$\epsilon > 0$ such that the set
$$\{ x \in A(\mathbb{Q}) \mid \hat{h}_{nt}(x) \leq \epsilon \}$$
is not Zariski dense in $W$.

Earlier, Zhang [Zha95a] had proved a similar result for the multiplicative
group $\mathbb{G}_m^m$. Zhang [Zha95b, Zha06] also proposed a more general conjecture
for what he called polarizable morphisms; a morphism $\Phi : X \rightarrow X$ on a
projective variety $X$ is said to be polarizable if there is an ample line bundle $L$
on $X$ such that $\Phi^* L \cong q L$ for some integer $q > 1$. When a polarizable
map $\Phi$ is defined over a number field, it gives rise to a canonical height $\hat{h}_\Phi$
with the property that $\hat{h}_\Phi(\Phi(\alpha)) = q \hat{h}_\Phi(\alpha)$ for all $\alpha \in X(\mathbb{Q})$. Zhang makes
the following Bogomolov-type conjecture in this more general context.

Conjecture 1.2. (Zhang) Let $\Phi : X \rightarrow X$ be a polarizable morphism of a
projective variety defined over a number field and let $W$ be a subvariety of
$X$ that is not preperiodic under $\Phi$. Then there exists an $\epsilon > 0$ such that the set
$$\{ x \in W(\mathbb{Q}) \mid h_\Phi(x) \leq \epsilon \}$$
is not Zariski dense in $W$.

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The definition of preperiodicity for varieties here is the same as the usual definition of preperiodicity for points. More precisely, for any quasiprojective variety \( X \), any endomorphism \( \Phi : X \rightarrow X \), and any subvariety \( V \subset X \), we say that \( V \) is \( \Phi \)-preperiodic if there exists \( N \geq 0 \), and \( k \geq 1 \) such that \( \Phi^N(V) = \Phi^{N+k}(V) \). Note that when \( A \) is an abelian variety and \( \Phi \) is a multiplication-by-\( n \) map (for \( n \geq 2 \)), an irreducible subvariety \( W \) is preperiodic if and only if it is a torsion translate of an abelian subvariety of \( A \).

We note that if \( X = (\mathbb{P}^1)^n \), and if \( \Phi \) is given by the coordinatewise action of \( z^d \) (for some \( d \geq 2 \)) on \( X \), then Conjecture 1.2 reduces to the result proved by Zhang in [Zha95a].

In this paper, we prove the following special case of Conjecture 1.2.

**Theorem 1.3.** Let \( f_1, \ldots, f_m \in \mathbb{Q}[X] \) be polynomials of degree \( d > 1 \), let \( \Phi := (f_1, \ldots, f_m) \) be their coordinatewise action on \( \mathbb{A}^m \), and let \( L \) be a line in \( \mathbb{A}^m \) defined over \( \mathbb{Q} \). If \( L \) is not \( \Phi \)-preperiodic, then there exists an \( \epsilon > 0 \) such that

\[
S_{L, \Phi, \epsilon} := \{ x \in L(\mathbb{Q}) \mid \hat{h}_{\Phi}(x) \leq \epsilon \}
\]

is finite (see Section 2 for the definition of \( \hat{h}_{\Phi} \)).

Baker and Hsia [BH05, Theorem 8.10] previously proved Theorem 1.3 in the special case where \( f_1 = f_2 \) and \( m = 2 \).

In Section 2 we introduce our notation for canonical heights associated to polynomials, then in Section 3 we present some general results regarding polynomials which share the same Julia set. In Section 4 we prove Theorem 1.3, and then describe few examples of preperiodic lines under the coordinatewise action of polynomials of same degree.

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## 2. Preliminaries

**Heights.** Let \( \mathbb{M}_\mathbb{Q} \) be the usual set of absolute values on \( \mathbb{Q} \), normalized so that the archimedean absolute value is simply the absolute value \( | \cdot | \) and \( |p|_p = 1/p \) for each \( p \)-adic absolute value \( | \cdot |_p \). For any finite extension \( K \) of \( \mathbb{Q} \) we define \( \mathbb{M}_K \) to be the set of absolute values on \( K \) that extend elements of \( \mathbb{M}_\mathbb{Q} \). Then, for any \( x \in \overline{\mathbb{Q}} \) we define the Weil height of \( x \) to be

\[
 h(x) = \frac{1}{[\mathbb{Q}(x) : \mathbb{Q}]} \sum_{v \in \mathbb{M}_\mathbb{Q}(x)} \sum_{w | v} \log \max\{ |x|_w^{[\mathbb{Q}(x)_w : \mathbb{Q}_v]}, 1 \}
\]

where \( \mathbb{Q}_v \) and \( \mathbb{Q}(x)_w \) are the completions of \( \mathbb{Q} \) and \( \mathbb{Q}(x) \) at \( v \) and \( w \) respectively (see [BG06, Chapter 1] for details).
For a polynomial \( f \in \mathbb{Q}[X] \) of degree greater than 1, define the \( f \)-canonical height \( \hat{h}_f : \mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0} \) by

\[
(2.1) \quad \hat{h}_f(x) = \lim_{n \to \infty} \frac{h(f^n(x))}{(\deg f)^n},
\]

following Call-Silverman [CS93] (where \( f^n \) denotes the \( n \)-th iterate of \( f \)).

Let \( f_1, \ldots, f_m \in \mathbb{Q}[X] \) be polynomials of degree \( d > 1 \), and let \( \Phi := (f_1, \ldots, f_m) \) be their coordinatewise action on \( \mathbb{A}^m \); that is,

\[
\Phi(x_1, \ldots, x_m) = (f_1(x_1), \ldots, f_m(x_m)).
\]

We define the \( \Phi \)-canonical height \( \hat{h}_\Phi : \mathbb{A}^m(\mathbb{Q}) \longrightarrow \mathbb{R}_{\geq 0} \) by

\[
\hat{h}_\Phi(x_1, \ldots, x_m) = \sum_{i=1}^m \hat{h}_{f_i}(x_i).
\]

Note that while \( \mathbb{A}^m \) is not a projective variety, \( \Phi \) extends to a map \( \tilde{\Phi} : (\mathbb{P}^1)^m \longrightarrow (\mathbb{P}^1)^m \). Furthermore, \( \tilde{\Phi} \) is polarizable, since

\[
\tilde{\Phi}^* \bigotimes_{i=1}^m \text{pr}_i^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \bigotimes_{i=1}^m \text{pr}_i^* \mathcal{O}_{\mathbb{P}^1}(d),
\]

where \( \text{pr}_i \) is the projection of \( (\mathbb{P}^1)^m \) onto its \( i \)-th coordinate.

**Remark.** Theorem 1.3 is *not* true if one allows the polynomials \( f_i \) to have different degrees. This is easily seen, for example, in the case where \( m = 2 \), the line \( L \) is the diagonal, and \( f_2 = f_1^2 \). The map \( \Phi = (f_1, \ldots, f_m) \) is only polarizable when \( \deg f_i = \deg f_j \), so this is not a counterexample to Conjecture 1.2.

### 3. Symmetries of the Julia set

In this section we recall the main results regarding polynomials which share the same Julia set. For a polynomial \( f \in \mathbb{C}[X] \), we let \( J(f) \) denote the Julia set of \( f \) (see [Bea91, Chapter 3] or [Mil99] for the definition of a Julia set of a rational function over the complex numbers). As proved by Beardon [Bea90, Bea92], any family of polynomials which have the same Julia set \( J \) is determined by the symmetries of \( J \).

**Definition 3.1.** If \( f \in \mathbb{C}[z] \) is a polynomial and \( J(f) \) is its Julia set, then the symmetry group \( \Sigma(f) \) of \( J(f) \) is defined by

\[
\Sigma(f) = \{ \sigma \in \mathcal{C} : \sigma(J(f)) = J(f) \},
\]

where \( \mathcal{C} \) is the group of conformal Euclidean isometries.

Beardon [Bea90, Lemma 3] computed \( \Sigma(f) \) for any \( f \in \mathbb{C}[z] \).

**Lemma 3.2.** The isometry group \( \Sigma(f) \) is a group of rotations about some point \( \zeta \in \mathbb{C} \), and it is either trivial, or finite cyclic group, or the group of all rotations about \( \zeta \).
To prove the result above, Beardon uses the fact that each polynomial $f$ of degree $d \geq 2$ is conjugate to a monic polynomial $\hat{f} \in \mathbb{C}[z]$ which has no term in $z^{d-1}$. Clearly, if $\hat{f} = \gamma \circ f \circ \gamma^{-1}$ where $\gamma \in \mathbb{C}[z]$ is a linear polynomial, then $J(\hat{f}) = \gamma(J(f))$; thus it suffices to prove Lemma 3.2 for $\hat{f}$ instead of $f$. If $\hat{f}(z) = z^d$, then $J(\hat{f})$ is the unit circle, and $\Sigma(\hat{f})$ is the group of all rotations about 0 (see also [Bea90, Lemma 4]). If $\hat{f}$ is not a monomial, then we choose $b \geq 1$ maximal such that we can find $a \geq 0$ and $\hat{f}_1 \in \mathbb{C}[z]$ satisfying $\hat{f}(z) = z^a \hat{f}_1(z^b)$. In this case, $J(\hat{f})$ is the finite cyclic group of rotations generated by the multiplication by $\exp(2\pi i/b)$ on $\mathbb{C}$ (see [Bea90, Theorem 5]). Beardon also proves the following result (see [Bea90, Lemma 7]) which we will use later.

**Lemma 3.3.** If $f \in \mathbb{C}[z]$ is a polynomial of degree $d \geq 2$ and $\sigma \in J(f)$, then $f \circ \sigma = \sigma^d \circ f$.

The following classification of polynomials which have the same Julia set is proven by Beardon in [Bea92, Theorem 1].

**Lemma 3.4.** If $f, g \in \mathbb{C}[z]$ have the same Julia set, then there exists $\sigma \in \Sigma(f)$ such that $g = \sigma \circ f$.

4. Proof of our main result

**Proof of Theorem 1.3.** Suppose that for every $\epsilon > 0$, the set $S_{L,\Phi,\epsilon}$ is infinite. We will show that this implies that $L$ must be $\Phi$-preperiodic.

We first note that it suffices to prove the theorem for the line $L' = (\sigma_1, \ldots, \sigma_m) \circ (L)$ and the map

$$\Phi' = (\sigma_1 \circ f_1 \circ \sigma_1^{-1}, \ldots, \sigma_m \circ f_m \circ \sigma_m^{-1})$$

for some linear automorphisms $\sigma_1, \ldots, \sigma_m$ of $\mathbb{A}^1$. This follows from the fact that $L$ is preperiodic for $\Phi$ if and only if $L'$ is preperiodic for $\Phi'$ along with the equality

$$\hat{h}_{\Phi'}(\sigma_1 \alpha_1, \ldots, \sigma_m \alpha_m) = \hat{h}_{\Phi}(\alpha_1, \ldots, \alpha_m).$$

Note that (4.1) is a simple consequence of Definition 2.1, since $|h(\sigma \cdot x) - h(x)|$ is bounded for all $x \in \overline{\mathbb{Q}}$.

We now proceed by induction on $m$; the case $m = 1$ is obvious.

If the projection of $L$ on any of the coordinates consists of only one point, we are done by the inductive hypothesis. Indeed, without loss of generality, assume the projection of $L$ on the first coordinate equals $\{z_1\}$, then $L = \{z_1\} \times L_1$, where $L_1 \subset \mathbb{A}^{m-1}$ is a line, and $\hat{h}_{f_1}(z_1) = 0$. Since only preperiodic points have canonical height equal to 0 (see [CS93, Cor. 1.1.1]), we conclude that $z_1$ is $f_1$-preperiodic, and thus we are done by the induction hypothesis applied to $L_1$.

Suppose now that $L$ projects dominantly onto each coordinate of $\mathbb{A}^m$. For each $i = 2, \ldots, m$, we let $L_i$ be the projection of $L$ on the first and the $i$-th coordinates of $\mathbb{A}^m$. Then $L_i$ is a line given by an equation $X_1 = \sigma_i(X_i)$,
for some linear polynomial $\sigma_i \in \mathbb{U}[X]$. Clearly, it suffices to show that for each $i = 2, \ldots, m$, the line $L_i$ is preperiodic under the action of $(f_1, f_i)$ on the corresponding two coordinates of $\mathbb{A}^m$. Indeed, if for each $i = 2, \ldots, m$, we show that there exist $a_i, b_i \in \mathbb{N}$ (with $a_i < b_i$) such that $(f_1^{a_i}, f_i^{b_i})(L_i) = (f_1^{a_i + b}, f_i^{b_i})(L_i)$, then

$$(f_1^{a_1}, \ldots, f_m^{a_m})(L) = (f_1^{a_1 + b}, \ldots, f_m^{a_1 + b})(L),$$

where $a := \max_{i=2}^m a_i$, and $b$ is the least common multiple of all $(b_i - a_i)$ for $i = 2, \ldots, m$.

Let $\bar{f}_i := \sigma_i \circ f_i \circ \sigma_i^{-1}$ and let $\Delta = (x, x) \in \mathbb{A}^2$ be the diagonal on $\mathbb{A}^2$. By our remarks at the beginning of the proof, it suffices to show that $(\text{id}, \sigma_i)(L_i) = \Delta$ is preperiodic under the action of $(f_1, \bar{f}_i)$. Furthermore, the fact that we have an infinite sequence $(z_{n,1}, z_{n,i}) \in L_i(\mathbb{Q})$ with

$$\lim_{n \to \infty} \hat{h}_{f_1}(z_{n,1}) = \lim_{n \to \infty} \hat{h}_{\bar{f}_i}(z_{n,i}) = 0$$

implies that we have

$$\lim_{n \to \infty} \hat{h}_{\bar{f}_i}(z_{n,1}) = 0,$$

because of (4.1). Fix an embedding $\theta : \mathbb{Q} \to C$ and let $f_1^\theta$ and $\bar{f}_i^\theta$ be the images of $f_1$ and $\bar{f}_i$, respectively, in $C[X]$ under this embedding. Then, by [BH05, Corollary 4.6], the Galois orbits of the points $\{z_{n,1}\}_{n \in \mathbb{N}}$ are equidistributed with respect to the equilibrium measures on the Julia sets of both $f_1^\theta$ and $\bar{f}_i^\theta$. Since the support of the equilibrium measure $\mu_g$ of a polynomial $g \in C[X]$ is equal to the Julia set of $g$ ([BH05, Section 4]), we must have $J(f_1^\theta) = J(f_i^\theta)$.

By Lemma 3.4 (see also [BE87, AH96]), there exists a conformal Euclidean symmetry $\mu_i : z \mapsto a_i z + b_i$ such that $\mu_i(J(f_1^\theta)) = J(f_i^\theta)$ and $\bar{f}_i^\theta = \mu_i \circ f_1^\theta$. Note that $a_i$ and $b_i$ must be in the image of $\mathbb{Q}$ under $\theta$ since the coefficients of $f_1^\theta$ and $f_i^\theta$ are. Let $\tau_i$ be the map $\tau_i : z \mapsto \theta^{-1}(a_i) z + \theta^{-1}(b_i)$. Then we have $\bar{f}_i = \tau_i \circ f_1$.

If $\tau_i$ has infinite order, then it follows from [Bea90, Lemma 4] that there exist linear polynomials $\gamma_1, \gamma_i$ such that $\gamma_1 \circ f_1 \circ \gamma_i^{-1} = \gamma_i \circ f_i \circ \gamma_i^{-1} = X^d$. In this case, we reduce our problem to the usual Bogomolov conjecture for $\mathbb{G}_m^2$, proved by Zhang [Zha92]. Indeed, Zhang proves that if a curve $C$ in $\mathbb{G}_m^2$ has an infinite family of algebraic points with height tending to zero, then it must be a torsion translate of an algebraic subgroup of $\mathbb{G}_m^2$; that is, $C = \xi A$ where $\xi$ has finite order and $A$ is an algebraic subgroup of $\mathbb{G}_m^2$. Since $(\xi A)^n = \xi^n A$ and $\xi$ has finite order, it is clear that such a curve is preperiodic under the map $(X, Y) \mapsto (X^d, Y^d)$.

We may suppose then that $\tau_i$ has finite order. By Lemma 3.3, we have $f_1 \circ \tau_i = \tau_i^d \circ f_1$. Thus, we have

$$\bar{f}_i^k = \tau_i^{(d^k-1)/(d-1)} \circ f_1^k$$
for all \( k \geq 1 \). Since \( \tau_i \) has finite order, we conclude that the set
\[
\{ \tau_i^{(d^k-1)/(d-1)} \}_{k \geq 0}
\]
is finite. This implies that the set of curves of the form \((f_1^k, \tilde{f}_1^k)(\Delta)\) is finite, which means the diagonal subvariety \( \Delta \) is preperiodic under the action of \((f_1, \tilde{f}_1)\), as desired. \( \square \)

4.1. Examples. Note that \( \Delta \) is periodic under \( \Phi \) only if there is some \( n \) such that \( f^n = g^n \). Take, for example, \( f(x) = x^3 \), \( g(x) = -x^3 \); then we have \( f^2 = g^2 \), so \( \Delta \) has period two under the action of \( \Phi \). More generally, there exists some \( n \) such that \( f^n = g^n \) if and only if \( f(x) = -\beta + \gamma h(x+\beta) \) and \( g(x) = -\beta + h(x+\beta) \) for some \( \gamma \in \mathbb{C}^* \), \( \beta \in \mathbb{C} \) and \( h \in x^r \mathbb{C}[x^s] \) (with \( r, s \geq 0 \)) such that \( \gamma^s = 1 \) and \( \gamma^{(d^n-1)/(d-1)} = 1 \) (see [GTZ08, Prop. 6.3]). On the other hand, if \( f(x) = h(x^s) \) for some positive integer \( s \geq 2 \) and \( \zeta \in \mathbb{C} \) is an \( s \)-th root of unity, and \( g(x) = \zeta \cdot f(x) \), then \( \Delta \) is \( \Phi \)-preperiodic, but it is not \( \Phi \)-periodic.

We believe that it is possible to extend the methods of the proof of Theorem 1.3 to the case of arbitrary rational maps \( \varphi_1, \ldots, \varphi_m \) of the same degree, though the proof seems to be much more difficult, requiring in particular Mimar's [Mim97] results on arithmetic intersections of metrized line bundles and an analysis of Douady-Hubbard-Thurston's [DH93] classification of critically finite rational maps with parabolic orbifolds. We intend to treat this problem in a future paper.

References


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