A REFORMULATION OF THE DYNAMICAL MANIN-MUMFORD CONJECTURE

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Abstract. We advance a new conjecture in the spirit of the Dynamical Manin-Mumford Conjecture. We show that our conjecture holds for all polarizable endomorphisms of abelian varieties and for all polarizable endomorphisms of \((\mathbb{P}^1)^N\). Furthermore, we show various examples which highlight the restrictions one would need to consider in formulating any general conclusion in the Dynamical Manin-Mumford Conjecture.

1. Introduction

As usual in algebraic dynamics, given a self-map \(\Phi: X \rightarrow X\) of a quasi-projective variety \(X\), we denote by \(\Phi^n\) the \(n\)-th iterate of \(\Phi\). Given a point \(x \in X\), we let \(O_{\Phi}(x) = \{\Phi^n(x): n \in \mathbb{N}\}\) be the orbit of \(x\). Recall that a point \(x\) is periodic if there exists some \(n \in \mathbb{N}\) such that \(\Phi^n(x) = x\); a point \(y\) is preperiodic if there exists \(m \in \mathbb{N}\) such that \(\Phi^m(y)\) is periodic.

One of the most fundamental questions in algebraic dynamics is the Dynamical Manin-Mumford Conjecture, originally formulated by Zhang in early 1990’s. Essentially, one expects the following to be true: given an endomorphism \(\Phi\) of a quasiprojective variety \(X\) defined over a field \(K\) of characteristic 0, then for any subvariety \(Z \subseteq X\), if \(Z\) contains a Zariski dense set of preperiodic points under the action of \(\Phi\), then \(Z\) is itself preperiodic, i.e., there exist integers \(0 \leq m < n\) such that \(\Phi^m(Z) = \Phi^n(Z)\) (or, at least, the Zariski closure of \(\Phi^m(Z)\) is the Zariski closure of \(\Phi^n(Z)\)). However, as noted by Zhang in \cite{Zha06}, one would need to impose an additional hypothesis on \(\Phi\) in order to rule out simple counterexamples coming from the coordinatewise action of different iterates of the same rational function acting on the coordinate axes of \((\mathbb{P}^1)^N\). So, Zhang \cite{Zha06} advanced the above conjecture in the case of polarizable endomorphisms \(\Phi\) of projective varieties \(X\) (i.e., there exists an ample line bundle \(L\) such that \(\Phi^*L\) is linearly equivalent with \(L^\otimes d\) for some \(d > 1\)).

However, there are counterexamples to the above conjecture using polarizable endomorphisms \(\Phi\) of a power of a complex multiplication (CM) elliptic curve \(E\) (for more details, see Section 3). In light of such counterexamples, an extra hypothesis in the Dynamical Manin-Mumford Conjecture was added (see the paper \cite{GTZ11}). More precisely, one expects that if \(Z\) contains a Zariski dense set of preperiodic points \(x\) with the property that the tangent subspace of \(Z\) at \(x\) is preperiodic under the induced action of \(\Phi\) on the Grassmanian \(\text{Gr}_{\dim(Z)}(T_{X,x})\), then \(Z\) should be itself preperiodic under the action of \(\Phi\).

However, the above extra hypothesis regarding the induced action of \(\Phi\) on the tangent subspace seems too strong; we still expect that the governing principle in the conjecture formulated originally by Zhang should hold (see also the discussion of the Dynamical Manin-Mumford Conjecture from \cite{YZ17} and also the results of \cite{DF17}, which hold beyond the case of polarizable endomorphisms). Therefore, we expect that if a subvariety \(Z\) of a projective variety \(X\) endowed with a polarizable endomorphism \(\Phi\) defined over a field \(K\) of characteristic 0 contains a Zariski dense set of preperiodic points, then either \(Z\) is itself preperiodic, or there exists an additional rigidity for the dynamical system, which may be expressed in the existence of an endomorphism \(\Psi\) under which \(Z\)
is preperiodic and moreover, $\Psi$ commutes with a suitable iterate of $\Phi$. Before properly stating our conjecture, we start with the definition of \textit{dynamically special} subvarieties with respect to a given polarizable endomorphism.

\textbf{Definition 1.1.} Let $\Phi$ be a polarizable endomorphism of a smooth projective variety $X$ defined over a field of characteristic 0. We say that $Z \subseteq X$ is a \textit{dynamically special} subvariety with respect to $\Phi$, if there exists a positive integer $n$, there exists a subvariety $Z \subseteq Y \subseteq X$ and there exists a polarizable endomorphism $\Psi$ of $X$ such that the following holds:

\begin{itemize}
  \item $Y$ is invariant under both $\Phi^n$ and $\Psi$;
  \item $\Phi^n \circ \Psi = \Psi \circ \Phi^n$ on $Y$; and
  \item $Z$ is preperiodic under the action of $\Psi$, i.e., there exist distinct positive integers $k$ and $\ell$ such that $\Psi^k(Z) = \Psi^\ell(Z)$.
\end{itemize}

The above Definition 1.1 is partially motivated by a similar phenomenon occurring in the arithmetic of Drinfeld modules regarding endomorphisms which admit suitable commuting iterates (see [Ghi05] and also, [Pin06] for the introduction of the so-called \textit{brother structure} in the context of Drinfeld modules).

\textbf{Conjecture 1.2.} Let $\Phi$ be a polarizable endomorphism of a projective smooth variety $X$ defined over a field $K$ of characteristic 0. Let $Z \subseteq X$ be a subvariety containing a Zariski dense set of preperiodic points. Then $Z$ is dynamically special with respect to $\Phi$.

In Section 2, we will prove that Conjecture 1.2 holds for polarizable endomorphisms of abelian varieties and also holds for polarizable endomorphisms of $(\mathbb{P}^1)^N$. We note that the aforementioned two cases were also the most studied cases of the Dynamical Manin-Mumford Conjecture (see [BH05, GT10, GTZ11, GNY18, GNY19]) and therefore, we are able to infer in our proofs some of the most difficult results previously established for the Dynamical Manin-Mumford question (such as the result from [GNY18] for endomorphisms of $(\mathbb{P}^1)^N$ given by the coordinatewise action of non-Lattés maps). We will prove our results each time under the assumption that the varieties in question are defined over a number field; most of our proofs extend easily to the function field case but we prefer at this (still) early stage in dealing with the Dynamical Manin-Mumford Conjecture to deal with the number field case alone (just as it was the case of the conjecture advanced by Zhang in his groundbreaking article [Zha06]).

Our paper grew from our search for finding the most natural conclusion in the Dynamical Manin-Mumford Conjecture, which would survive the various counterexamples we knew existed for the earlier version of this conjecture. We were also encouraged in our efforts by Shouwu Zhang’s question that he posed to us for formulating a characterization of the class of dynamically special subvarieties in the Dynamical Manin-Mumford Conjecture which would be closed under taking intersections. Unfortunately, we could not find such a characterization and we will present in Section 3 various counterexamples in this direction. In particular, our various examples from Section 3 show the difficulty one encounters in formulating a satisfying conclusion in the Dynamical Manin-Mumford Conjecture which would take into account all possible constructions.

\section{Proofs}

We start with an easy standard lemma.

\textbf{Lemma 2.1.} Let $\Phi$ and $\Psi$ be commuting polarizable endomorphisms of a smooth projective variety $X$ defined over a number field $K$. Let $x \in X(K)$ be a preperiodic point of $\Phi$. Then $x$ is a preperiodic point for $\Psi$.

\textbf{Proof.} So, we know there exist integers $0 \leq m < n$ such that $\Phi^m(x) = \Phi^n(x)$. We let $\widehat{h}_\Phi : X(K) \to \mathbb{R}_{\geq 0}$ be the canonical height constructed with respect to the polarizable endomorphism
Let \( \Phi \) be a polarizable endomorphism of the abelian variety \( X \). Proof of Conjecture 1.2 for polarizable endomorphisms of abelian varieties defined over number fields.

We know that the polarizable endomorphism \( \Phi \) of \( (\mathbb{P}^1)_N \) is dynamically special. In this case, for each \( m \), \( \Phi \) and \( \Phi_m \) are commuting endomorphisms of \( X \). Then Raynaud’s theorem \([\text{Ray83}]\) yields that \( Z \) is a torsion translate of an algebraic subgroup of \( X \). Therefore \( X \) is preperiodic under the multiplication-by-2 map (which commutes with \( \Phi \)) and so, \( Z \) is dynamically special.

Using \([\text{GNY18}]\) (which actually builds on the work from \([\text{GNY19}]\) along with a more careful analysis of the case of polarizable endomorphisms of powers of \( CM \) elliptic curves, we can prove Conjecture 1.2 for all polarizable endomorphisms of \((\mathbb{P}^1)_N\).

\textbf{Proof of Conjecture 1.2 for polarizable endomorphisms of \((\mathbb{P}^1)_N\) defined over number fields.} So, we assume that \( \Phi \) is preperiodic under the action of \( \Phi \) and \( \Phi_m \) are commuting endomorphisms of \( X \). Then Raynaud’s theorem \([\text{Ray83}]\) yields that \( Z \) is a torsion translate of an algebraic subgroup of \( X \). Therefore \( X \) is preperiodic under the multiplication-by-2 map (which commutes with \( \Phi \)) and so, \( Z \) is dynamically special.

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\textbf{Proof of Conjecture 1.2 for polarizable endomorphisms of \((\mathbb{P}^1)_N\) defined over number fields.} So, we assume that \( \Phi \) is preperiodic under the action of \( \Phi \) and \( \Phi_m \) are commuting endomorphisms of \( X \). Then Raynaud’s theorem \([\text{Ray83}]\) yields that \( Z \) is a torsion translate of an algebraic subgroup of \( X \). Therefore \( X \) is preperiodic under the multiplication-by-2 map (which commutes with \( \Phi \)) and so, \( Z \) is dynamically special.

Next we study \( Z_2 \), which we know contains a Zariski dense set of periodic points under the coordinatewise action of \((N-m) \) Latté maps. In this case, for each \( i = m+1, \ldots, N \), the Latté map \( f_i \) corresponds to an elliptic curve \( E_i \) equipped with an endomorphism \( g_i \) along with a projection map \( \pi_i : E_i \rightarrow \mathbb{P}^1 \) such that \( \pi_i \circ g_i = f_i \circ \pi_i \). We let \( A := \prod_{i=m+1}^N E_i \) and let \( g \) be the endomorphism of \( A \) given by the coordinatewise action of each \( g_i : E_i \rightarrow E_i \); also, we let \( \pi : A \rightarrow (\mathbb{P}^1)^{N-m} \) be the projection map induced by each coordinatewise projection \( \pi_i : E_i \rightarrow \mathbb{P}^1 \). Then \( W := \pi^{-1}(Z_2) \) is a subvariety of \( A \) containing a Zariski dense set of periodic points under the action of \( \Phi \) and so, by Lemma 2.1, we obtain that each irreducible component of \( W \) is a torsion translate of an algebraic subgroup of \( A \). In particular, a given irreducible component \( W_2 \) of \( W \) which projects...
dominantly onto $Z_2$ (through $\pi$) is of the form $\zeta + H$, where $\zeta \in A_{\text{tor}}$ and $H$ is a (connected) algebraic subgroup of $A$. So, $W_2$ is preperiodic under the action of the multiplication-by-$[d]$ map on $A$, which also commutes with $g$. Therefore, the Latté maps $h_i : \mathbb{P}^1 \to \mathbb{P}^1$ induced by the multiplication-by-$d$ maps on each $E_i$ commute with $f_i$ for each $i = m + 1, \ldots, N$; furthermore, $Z_2$ is preperiodic under the induced action $\Psi_2$ on the last $(N - m)$ coordinate axes of $(\mathbb{P}^1)^N$ given by the coordinatewise action of the Latté maps $g_i$. Also, note that $\deg(g_i) = d^2$ for each $i = m + 1, \ldots, N$.

Now, we let $\Psi_1 := \Phi_1^2$, i.e., the coordinatewise action given by $f_1^2$ on the first $m$ coordinate axes of $(\mathbb{P}^1)^N$. Then $\Psi := \Psi_1 \times \Psi_2$ is a polarizable endomorphism of $(\mathbb{P}^1)^N$ (of degree $d^2$), commuting with $\Phi$. Furthermore, $Z$ is preperiodic under the action of $\Psi$, as desired. \qed

3. Examples

In [GTZ11], examples are given of polarizable morphisms $\Phi : X \to X$ with subvarieties $Z \subset X$ such that $Z$ contains a Zariski dense set of (pre)periodic points under the action of $\Phi$ but $Z$ is not preperiodic under $\Phi$. One such example is given by taking $E$ to be an elliptic curve such that $\text{End}(E) = \mathbb{Z}[i]$ and letting $X = E \times E$ with $\Phi = ([3 + 4i, 3 - 4i])$ (that is, the map that acts as multiplication-by-$[3 + 4i]$ in the first coordinate and multiplication-by-$[3 - 4i]$ in the second coordinate) with $Z$ the diagonal in $E \times E$. One can easily see then that not only does $Z$ contain a Zariski dense set of preperiodic points under $\Phi$ but it also contains a Zariski dense set of periodic points under $\Phi$. We note that of course $Z$ is invariant under $(f, f)$ for any $f \in \text{End}(E)$.

Since in the classical case of the Manin-Mumford conjecture, the special subvarieties (i.e., the ones containing a Zariski dense set of torsion points of a given abelian variety $X$) are the torsion translates of $X$, then the intersection of two such special subvarieties is once again special. So, it was a very natural question posed by Shouwu Zhang whether a similar conclusion might be achieved in the dynamical setting, i.e., intersection of two dynamically special subvarieties (with a proper definition) would be again dynamically special. The following example shows that the intersection of preperiodic subvarieties (under the action of a polarizable endomorphism) does not have to be itself a preperiodic subvariety; furthermore, it does not even have to contain a Zariski dense set of preperiodic points.

**Example 3.1.** Let $\Phi : \mathbb{P}^2 \to \mathbb{P}^2$ by $\Phi([x : y : z]) = [x^2 + z^2, y^2 + z^2 : z^2]$ and let $Z_1$ be the curve given by $x = y$. Then $\Phi^{-1}(Z_1)$ is the union of $Z_1$ with the curve $x = -y$, which we call $Z_2$. So, both $Z_1$ and $Z_2$ are preperiodic under the action of $\Phi$. However, the intersection $Z_1 \cap Z_2$ is the point $[0 : 0 : 1]$, which is not preperiodic under $\Phi$. \hfill \diamond

Example 3.1 shows that one cannot formulate a characterization of the dynamically special subvarieties in the conclusion of the Dynamical Manin-Mumford Conjecture involving preperiodic points and still expect that the intersection of two dynamically special subvarieties would be dynamically special. Note that Lemma 2.3 would yield that a non-preperiodic point for a polarizable endomorphism $\Phi$ would remain non-preperiodic under any polarizable endomorphism $\Psi$ commuting with $\Phi$. Another reasonable problem would be to consider subvarieties containing a Zariski dense set of periodic points, especially in the light of the following conjecture we propose.

**Conjecture 3.2.** Let $\Phi$ be a polarizable endomorphism of a projective smooth variety $X$ defined over a field of characteristic 0. If $Z \subset X$ contains a Zariski dense set of preperiodic points, then there exists $\ell \in \mathbb{N}$ such that $\Phi^\ell(Z)$ contains a Zariski dense set of periodic points.

We note that if the dynamical system $(X, \Phi)$ from Conjecture 3.2 were to satisfy the conclusion from the original Dynamical Manin-Mumford Conjecture formulated by Zhang, then we would know that there exists $\ell \in \mathbb{N}$ such that $\Phi^\ell(Z)$ is periodic under the induced action of $\Phi^\ell$. Then, using Fakhruddin’s theorem [Fak03] regarding the existence of a Zariski closed set of periodic points for any polarizable dynamical system, we would obtain the desired conclusion in Conjecture 3.2.
In particular, the conclusion of Conjecture \[3.2\] is known for polarizable endomorphisms of \((\mathbb{P}^1)^N\) which are not given by coordinatewise action of Lattés maps (see [GNY13, Theorem 1.1]). Employing Raynaud’s theorem [Ray83], one can easily deduce that Conjecture \[3.2\] holds for polarizable endomorphisms of abelian varieties. A similar analysis as in our proof of Conjecture \[1.2\] for the case of all polarizable endomorphisms of \((\mathbb{P}^1)^N\) shows that Conjecture \[3.2\] also holds for arbitrary polarizable endomorphisms of \((\mathbb{P}^1)^N\).

On the other hand, as shown in the following example, the polarizability condition is crucial in Conjecture \[3.2\].

**Example 3.3.** This is an example of a surjective endomorphism \(\Phi\) of an affine variety \(X\) (defined over a field of characteristic 0) with the property that there exists a Zariski dense set of preperiodic points, but on the other hand, the periodic points of \(\Phi\) are not Zariski dense in \(X\); in particular, this justifies the polarizability assumption from Conjecture \[3.2\] (since of course we have \(\Phi^n(X) = X\) for all \(X\)). We let \(\Phi\) be the endomorphism of \(X = \mathbb{A}^2\) given by \(\Phi(x, y) = (x + y - 1, y^2)\). Then each point of the form \((x, \zeta_{2^n})\) is a preperiodic point for \(\Phi\) (for any \(x\) and any \(2^n\)-th root of unity \(\zeta_{2^n}\)). On the other hand, the only periodic points for \(\Phi\) are the points fixed by \(\Phi\), which are of the form \((x, 1)\). \(\diamondsuit\)

Now, we return to the question of finding a suitable property characterizing the dynamically special subvarieties so that intersections of such varieties would still satisfy the same property. The next example shows that even the intersection of two subvarieties that are periodic under a polarizable morphism \(\Phi\) need not contain a Zariski dense set of periodic points under \(\Phi\) (though it will contain a dense set of preperiodic points).

**Example 3.4.** Let \(E\) be an elliptic curve, let \(X = E \times E\), let \(\Phi(x, y) = ([3]x, [3]y)\), let \(Z_1\) be the diagonal in \(E \times E\), and let \(Z_2\) be the set of all points \((x, [4]x)\) where \(x \in E\). Then \(Z_1\) and \(Z_2\) are both invariant under \(\Phi\) but \(Z_1 \cap Z_2\) contains only points of form \((z, z)\) where \(z\) is a 3-torsion point of \(E\); all of these are strictly preperiodic under \(\Phi\) except for \((0, 0)\) (where 0 is the identity element of \(E\)). \(\diamondsuit\)

Note that in Example \[3.4\], all of the points in \(Z_1 \cap Z_2\) are however periodic under \(\Psi\) where \(\Psi(x, y) = ([m]x, [m]y)\) for any \(m\) that is not divisible by 3. In particular, this also motivated our Definition \[1.1\] for dynamically special subvarieties, which allow in their characterization another polarizable endomorphism \(\Psi\) commuting with a suitable iterate of \(\Phi\). Unfortunately, even this formulation is not enough for being preserved under taking intersections, as the following (more elaborate) example will show it.

**Example 3.5.** Let \(E\) be a (non-CM) elliptic curve and let \(f(z)\) be a polynomial of degree 25, which is disintegrated, according to the definition from [MS14] (i.e., \(f(z)\) is conjugated neither to \(z^{25}\) nor to \(C_{25}\), where for each \(n \in \mathbb{N}\), \(C_n\) is the \(n\)-th Chebyshev polynomial, which is the unique polynomial satisfying \(C_n(x + 1/x) = x^n + 1/x^n\)). Then, as shown in [GN16, Proposition 2.3], each polynomial commuting with an iterate of \(f(z)\) must have degree a power of 5.

Now, let \(X := E^2 \times (\mathbb{P}^1)^2\) and let \(\Phi\) be the endomorphism of \(X\) given by

\[
(w_1, w_2, z_1, z_2) \mapsto ([5]w_1, [5]w_2, f(z_1), f(z_2)).
\]

We let \(\Gamma_f \subset (\mathbb{P}^1)^2\) be the graph of the polynomial \(f\). We let \(\Delta \subset E^2\) be the diagonal (i.e., the set of all points \((w, w) \in E^2\); also, we let \(\Delta_6 \subset E^2\) be the set of all points of the form \((w, [6]w)\). Then we let \(Z_1 := \Delta \times \Gamma_f\) and \(Z_2 := \Delta_6 \times \Gamma_f\). Both \(Z_1\) and \(Z_2\) contain a Zariski dense set of periodic points under the action of \(\Phi\) (they are actually both invariant under the action of \(\Phi\)).

Now, \(Z := Z_1 \cap Z_2\) is the set \((\Delta \cap E[5]^2) \times \Gamma_f\), i.e., it consists of all points of the form \((w_5, w_5, z, f(z))\).
where \( w_5 \in E[5] \) is a 5-torsion point of \( E \), while \( z \) is an arbitrary point in \( \mathbb{P}^1 \).

Our choice for a disintegrated polynomial \( f(z) \) of degree 25 yields that any polarizable endomorphism \( \Psi \) of \( X \) which commutes with some iterate of \( \Phi \) needs to have degree equal to a power of 5. However, then the induced polarizable endomorphism \( \Psi_1 \) of \( E^2 \) also has its degree equal to a power of 5 and therefore, the points of the form \((w_5, w_5)\) (for \( w_5 \neq 0 \)) are strictly preperiodic under the action of \( \Psi_1 \).

\[ \diamond \]

The examples from this Section simply show the difficulty one encounters in trying to formulate the most satisfying conclusion in the Dynamical Manin-Mumford Conjecture. From all the examples we saw, it seems to us that Conjecture \([1.2]\) is the most natural conclusion one could expect for this difficult problem.

References


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