Test 1  Duration: 50 minutes
This test has 5 questions on 7 pages, for a total of 25 points.

• All questions are long-answer; you should give complete arguments and explanations for all your solutions; answers without justifications will not be marked.

• Continue on the back of the page if you run out of space.

• This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ___________________  Last Name: ___________________

Student-No: __________________

Signature: ___________________

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Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

   (i) speaking or communicating with other examination candidates, unless otherwise authorized;

   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;

   (iii) purposely viewing the written papers of other examination candidates;

   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,

   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
5 marks 1. Let \( n, k \in \mathbb{N} \) with \( n \geq 2 \). Show that

\[
(n - 1)^2 \mid (n^k - 1)
\]

if and only if

\[
(n - 1) \mid k.
\]

**Solution:** The solution is almost identical with the solution to Problem 2 from Practice Set 1.
2. Let $1 \leq a < b$ be positive integers. Prove that

$$(b^{2019} - a^{2019}) \nmid (a^{2019} + b^{2019}).$$

**Solution:** The solution is almost identical with the solution to Problem 3 from Practice Set 1.
3. Let $1 \leq x < y$ be positive integers. Show that if 

$$ \sqrt{x} - \sqrt{y} \in \mathbb{Z} $$

then we must have that both $\sqrt{x} \in \mathbb{Z}$ and $\sqrt{y} \in \mathbb{Z}$.

**Solution:** Assume $\sqrt{x} - \sqrt{y} = m \in \mathbb{Z}$. Then squaring both sides yields that 

$$ x + y - 2\sqrt{xy} = m^2 $$

and so, $\sqrt{xy}$ must be a rational number.

**Claim.** If $a$ is a positive integer with the property that $\sqrt{a}$ is a rational number, then actually $\sqrt{a}$ is an integer, i.e., $a$ is a perfect square.

**Proof of Claim.** Assume $\sqrt{a} = \frac{b}{c}$ with $b, c \in \mathbb{N}$ and furthermore, assume $\gcd(b, c) = 1$ and that $c > 1$ (i.e., assume that $\sqrt{a}$ is not an integer). But then $a = \frac{b^2}{c^2}$ and so, $ac^2 = b^2$, which contradicts the Fundamental Theorem of Arithmetic since the left-hand side is divisible by primes dividing $c$ which do not appear in the prime power factorization of $b$ (or of $b^2$). This contradiction yields that $c$ must be equal to 1, i.e., $\sqrt{a}$ must be an integer.

So, using this Claim, we conclude that $xy$ is a perfect square, say $xy = z^2$ for some positive integer $z$. But then the following is an integer:

$$ \sqrt{x} - \sqrt{y} = \sqrt{x} - \frac{z}{\sqrt{x}} = \frac{x - z}{\sqrt{x}} $$

and so, we get that also $\sqrt{x}$ must be a rational number (note that - very importantly - $x - z \neq 0$ since otherwise we would have $x = z = \sqrt{xy}$ and so, $x = y$, which is a contradiction). One more application of the above Claim yields that $\sqrt{x}$ must be an integer and since $\sqrt{x} - \sqrt{y}$, then also $\sqrt{y}$ is an integer, as desired.

This solution is similar to the solution to Problem 25 from Practice Set 1.
4. Let $a > 1$ be an integer. We define the sequence $\{x_n\}$ given by $x_1 = \sqrt[3]{a}$ and then $x_{n+1} = \sqrt[3]{ax_n}$ for all integers $n \geq 1$. Find all such integers $a$ with the property that $x_{2019}$ is a positive integer.

**Solution:** The solution is very similar to the solution of Problem 24 from Practice Set 1. The only difference is that in this case, we have that

$$x_n = a^{3^{-n-1}}.$$ 

We have that $\frac{3^n - 1}{2 \cdot 3^n} = \frac{b_n}{3^n}$ for some positive integer $b_n$, which is not divisible by 3 (since $3^n - 1$ is an even positive integer, not divisible by 3). But then

$$a^{b_{2019}}$$

is a positive integer

if and only if $a = c^{3^{2019}}$ for some positive integer $c$.

This last claim holds because for any two coprime positive integers $k$ and $\ell$, if $\sqrt[\ell]{a^k}$ is a positive integer (say, $z$), then it must be that $a = c^\ell$ for some positive integer $c$. Indeed, we would have that

$$a^k = z^\ell$$

and then using the Fundamental Theorem of Arithmetic along with inspection of the exponents of the prime powers in the left-hand side we get that each such exponent must be divisible by $\ell$ because this is the case for the exponents of the primes appearing in $z^\ell$; finally, using that $\gcd(k, \ell) = 1$, we conclude that each exponent appearing in the prime power factorization of $a$ must be divisible by $\ell$, i.e., $a = c^\ell$ for a positive integer $c$. 

5 marks 5. Let \( n > 1 \) be an integer. Prove that

\[
\sum_{i=1}^{n} \frac{1}{2i + 1}
\]

is not a positive integer.

**Solution:** The solution is similar to the solution to *Problem 18* from *Practice Set 1*.

We write in lowest terms, i.e.,

\[
\sum_{i=1}^{n} \frac{1}{2i + 1} = \frac{a}{b}
\]

with \( a, b \in \mathbb{N} \) coprime integers. We will prove that \( 3 \mid b \), which therefore proves that \( \frac{a}{b} \) is not an integer, as claimed.

We let \( k \) be a positive integer such that

\[
3^k \leq 2n + 1 < 3^{k+1}.
\]

Note that since \( n \geq 1 \) then \( 2n + 1 \geq 3 \) and so, indeed \( k \geq 1 \). We also note that - obviously - there is precisely one positive integer \( k := k(n) \) satisfying (2) for any given positive integer \( n \). Thus, for

\[
\sum_{i=1}^{n} \frac{1}{2i + 1}
\]

we encounter the fraction

\[
\frac{1}{3^k}
\]

in the sum

\[
\text{(4)}
\]

since \( 3^k \leq 2n + 1 \) (by (2)) and also because \( 3^k \) is an odd integer, and therefore it can be written as \( 2i + 1 \). On the other hand, because \( 2 \cdot 3^n \) is even, then the fraction

\[
\frac{1}{2 \cdot 3^k}
\]

does not appear in the sum (3).

Also, using (2), there is no fraction in the sum (3) which has its denominator divisible by \( 3^{k+1} \).

Combining (2), (4) and (5), we conclude that in the sum (3), there is precisely one fraction which has a denominator divisible by the power \( 3^k \) (and that fraction is actually \( \frac{1}{3^k} \)); except that fraction, all other fractions have their denominators divisible by a power of 3 which is at most \( 3^{k-1} \). In conclusion, the sum of all other fractions appearing in (3) (except \( \frac{1}{3^k} \)) can be written in lowest terms as

\[
\left( \sum_{i=1}^{n} \frac{1}{2i + 1} \right) - \frac{1}{3^k} = \frac{c}{d}
\]

for some positive coprime integers \( c \) and \( d \); furthermore the highest power of 3 appearing in \( d \) is \( 3^\ell \) for some non-negative integer \( \ell < k \). In particular, we can write

\[
d = 3^\ell \cdot d_1,
\]

\[
\text{(7)}
\]
for some positive integer $d_1$ not divisible by 3. So, using (6) and (1), we get

$$\frac{c}{d} + \frac{1}{3^k} = \frac{a}{b};$$

(8)

furthermore, the information regarding the power of 3 appearing in $d$ (along with (7)) yields that

$$\frac{c}{d} + \frac{1}{3^k} = \frac{3^{k-\ell}c + d_1}{3^{k-\ell}d}. $$

(9)

Because $k - \ell \geq 1$ and $d_1$ is not divisible by 3, we get that

$$3^{k-\ell}c + d_1$$

is not divisible by 3.

(10)

Combining (8) and (9), we obtain

$$b \cdot (3^{k-\ell} + d_1) = 3^{k-\ell}d \cdot a. $$

(11)

Equations (10) and (11) yield that $3 \mid b$, as claimed and so,

$$\sum_{i=1}^{n} \frac{1}{2i + 1}$$

is not a positive integer (see (1)).