1. Problems

Problem 1. Let $\alpha_1, \ldots, \alpha_r \in \mathbb{R}$. For each $\epsilon > 0$ prove that there exist infinitely many positive integers $n$ with the property that for each $i = 1, \ldots, r$ we have either $\{n\alpha_i\} < \epsilon$ or $\{n\alpha_i\} > 1 - \epsilon$. (By $\{x\}$ we always denote the fractional part of the real number $x$.)

Problem 2. Prove that the sum of 5 consecutive perfect squares is never a perfect square.

Problem 3. Let $n \in \mathbb{N}$. Prove that at least one of the following numbers is even:
\[
\left[ 2^n \cdot \sqrt{2} \right], \left[ 2^{n+1} \cdot \sqrt{2} \right], \ldots, \left[ 2^{2n} \cdot \sqrt{2} \right],
\]
where $[y]$ is always the integer part of the real number $y$.

Problem 4. Let $x \in \mathbb{R}$ be larger than 1 with the property that there exist infinitely many positive integers $n$ such that $n\sqrt{n} \in \mathbb{N}$. Prove that $x \in \mathbb{N}$. (As always, $[z]$ represents the integer part of the real number $z$.)

Problem 5. Let $x \in \mathbb{C} \setminus \{1\}$. If there exist three consecutive positive integers $n$ such that $x^n - 1 = x^{n-1} \in \mathbb{Z}$, then for each $n \in \mathbb{N}$, $\left[ \frac{x^n - 1}{x - 1} \right] \in \mathbb{Z}$.

Problem 6. Show that there exist infinitely many triples $(x, y, z)$ satisfying the equation $x^2 + y^2 = 2z^2 + 8$.

Problem 7. Show that there exists no prime number $p$ such that $3^p + 19(p - 1)$ is a perfect square.

Problem 8.
(i) Show that there exist infinitely many integers $x$, $y$ and $z$ such that $x^2 + y^2 = 2z^2 + 8$.
(ii) Show that there exist infinitely many integers $a, b, c$ such that $a^2 + b^2 = c^2 + 3$.

Problem 9. Find all positive integers $x$, $y$ and $z$ knowing that $x$ is prime and $x^2 + y^2 - 33z^2 = 8yz$.

Problem 10. Let $p$ be an odd prime number. Find all $a, b \in \mathbb{N}$ such that $a^2 + pa = b^2$. 
Problem 11. Let \(a, b, c \in \mathbb{Z}\) with \(a \neq 0\), and let \(f(x) = ax^2 + bx + c\) be the corresponding quadratic polynomial.

(a) Show that either \(b^2 - 4ac = 0\), or the set of prime numbers \(p\) for which there exists some \(n \in \mathbb{Z}\) such that \(p \mid f(n)\) and \(p \mid f'(n)\) is finite.

(b) Prove that the set of prime numbers \(p\) for which there exists some \(n \in \mathbb{Z}\) such that \(p \mid f(n)\) is infinite.

(c) If \(f(n)\) is a perfect square for each \(n \in \mathbb{Z}\) prove that there exists a polynomial \(g \in \mathbb{Z}[x]\) such that \(f(x) = (g(x))^2\) for all \(x\).

Problem 12. Let \(f(x) = x^3 + ax + b \in \mathbb{Z}[x]\) be a cubic polynomial and let \(r_1, r_2, r_3 \in \mathbb{C}\) be the roots of
\[
f(x) = 0.
\]
If \(r_1 = r_2\), then prove that \(r_1, r_2, r_3 \in \mathbb{Z}\).

Problem 13. Let \(f(x) = x^3 + ax + b \in \mathbb{Z}[x]\) be a cubic polynomial and let \(r_1, r_2, r_3 \in \mathbb{C}\) be the roots of
\[
f(x) = 0.
\]
If \(r_1 = r_2\), then prove that there exist infinitely many positive integers \(n\) such that \(f(n)\) is not a perfect square.

Problem 14. Let \(f(x) \in \mathbb{Z}[x]\), let \(n_0, \ell \in \mathbb{Z}\) and let \(p\) be a prime number. Assume the following:

- \(p \nmid \ell\)
- \(p^2 \mid (f(n_0 + \ell p) - f(n_0))\).

Then prove that \(p \mid f(n_0)\) (where \(f'(x)\) is the derivative of the polynomial \(f(x)\)).

Problem 15. Let \(f, g \in \mathbb{Z}[x]\) be polynomials with the property that there exists no \(r \in \mathbb{C}\) such that
\[
f(r) = g(r) = 0.
\]
Then prove that there exists a nonzero integer \(m\) and there exist \(P, Q \in \mathbb{Z}[x]\) such that
\[
f(x) \cdot P(x) + g(x) \cdot Q(x) = m.
\]

Problem 16. Let \(f \in \mathbb{Z}[x]\) be a nonconstant polynomial. Prove that the set
\[
S = \{p \text{ prime: there exist infinitely many positive integers } n \text{ such that } p \mid f(n)\}
\]
is infinite.

Problem 17. Let \(f \in \mathbb{Z}[x]\) be a nonconstant polynomial with the property that all the roots (in \(\mathbb{C}\)) for the equation
\[
f(x) = 0
\]
are distinct. Prove that there exist infinitely many positive integers \(n\) such that \(f(n)\) is not a perfect square.

Problem 18. Let \(f(x) \in \mathbb{Z}[x]\) be a polynomial of degree 3. Prove that there exist infinitely many positive integers \(n\) such that \(f(n)\) is not a perfect square.

Problem 19. Let \(f(x) = x^3 + 23\). Find all integers \(n\) with the property that \(f(n)\) is a perfect square.
2. Solutions

Problem 1. It suffices to prove the seemingly weaker statement that for each \( \epsilon > 0 \) there exists one positive integer \( n \) such that for each \( i = 1, \ldots, r \) either \( \{n \alpha_i\} < \epsilon \) or \( \{n \alpha_i\} > 1 - \epsilon \). Indeed, here’s how the “weaker” statement implies the “stronger” statement.

Let \( \epsilon > 0 \) and assume we found the integers \( n_1 < \cdots < n_k \) such that for each \( j = 1, \ldots, k \) we have that either \( \{n_j \alpha_i\} < \epsilon \) or \( \{n_j \alpha_i\} > 1 - \epsilon \).

We’ll construct next an integer \( n_{k+1} \) larger than all of these first \( k \) integers and sharing the same property those numbers have (relative to the \( \alpha_i \)’s). By the “weaker” statement above we can find a positive integer \( n \) such that for each \( i = 1, \ldots, r \) either \( \{n \alpha_i\} < \epsilon \) or \( \{n \alpha_i\} > 1 - \epsilon \), as desired.

Problem 2. Assume that there exist integers \( m \) and \( n \) such that
\[
(m - 2)^2 + (m - 1)^2 + m^2 + (m + 1)^2 + (m + 2)^2 = n^2.
\]
We would then have that
\[
5m^2 + 10 = n^2
\]
which yields that \( 5 \mid n \). We let \( n = 5n_1 \) for some integer \( n_1 \) and therefore
\[
m^2 + 2 = 5n_1^2.
\]
So,
\[
m^2 \equiv -2 \pmod{5}
\]
which is impossible: the nonzero quadratic residues modulo 5 are only 1 and 4 (not 3).

Problem 3. Assume all those numbers are indeed odd. So,
\[
2^n \sqrt{2} = 2x + 1 + r,
\]
where \( x \in \mathbb{N} \) and \( r = \{2^n \sqrt{2}\} \) is its fractional part, i.e., \( 0 < r < 1 \). Then
\[
2^{n+1} \sqrt{2} = 4x + 2 + 2r
\]
and using that the integer part \( \lfloor 2^{n+1} \sqrt{2} \rfloor \) is still odd, we deduce that
\[
1 \leq 2r < 2.
\]
Claim. For each \( k = 1, \ldots, n \) we have that
\[
0 < 1 - r < \frac{1}{2^k}.
\]
Proof of Claim. We already proved this inequality for \( k = 1 \). We argue by induction on \( k \). So, we have that for some \( k < n \):
\[
2^{n+k} \sqrt{2} = 2^{k+1} x + 2^{k+1} - 1 + (1 - 2^k(1 - r)),
\]
where \( [2^{n+k} \sqrt{2}] = 2^{k+1}x + 2^{k+1} - 1 \) is indeed odd, while 
\[
\{2^{n+k} \sqrt{2}\} = 1 - 2^k(1 - r).
\]

Then 
\[
2^{n+k+1} \sqrt{2} = 2^{k+2}x + 2^{k+2} - 1 + (1 - 2^{k+1}(1 - r)),
\]

where 
\[
0 < 2^{k+1}(1 - r) \leq 2.
\]

If \( 2^{k+1}(1 - r) > 1 \), then \( [2^{n+k} \sqrt{2}] = 2^{k+2}x + 2^{k+2} - 2 \), contradicting thus our assumption that all these integer parts are odd. So, as desired, we conclude that
\[
0 < 2^{k+1}(1 - r) \leq 1.
\]

So, letting \( s := 1 - r \), we have
\[
2^n \sqrt{2} = 2x + 2 - s,
\]
where \( 0 < s \leq 2^{-n} \). We also let \( y := x + 1 \), and hence
\[
0 < s = 2y - 2^n \sqrt{2} \leq 2^{-n}
\]
and
\[
0 < 2y + 2^n \sqrt{2} < 1 + 2^{n+1} \sqrt{2}.
\]

Therefore
\[
0 < 4y^2 - 2^{2n+1} < 2^{-n} + 2 \sqrt{2} < \frac{1}{2} + 3 < 4.
\]

However, \( 4 \mid 4y^2 - 2^{2n+1} \), and therefore we obtained a contradiction which shows that indeed one of the integer parts of those numbers must be even.

**Problem 4.** Assume \( x \not\in \mathbb{N} \). Then \( [x] < x < [x] + 1 \). Furthermore, for each \( n \in \mathbb{N} \) we have
\[
[x]^n < x^n < ([x] + 1)^n.
\]

Now, let \( n \in \mathbb{N} \) such that \( \sqrt{x^n} \in \mathbb{N} \). Since
\[
[x]^n \leq [x^n] < ([x] + 1)^n,
\]
and \( [x^n] \) must be a perfect \( n \)-th power, we conclude that \( [x^n] = [x]^n \).

We know there exist then infinitely many positive integers \( n \) such that \( [x^n] = [x]^n \). Let \( r \) be the fractional part of \( x \); clearly by our assumption, \( r > 0 \). So,
\[
x^n = ([x] + r)^n,
\]
and if \( n > \frac{1}{2} \) then (using also that \( [x] \geq 1 \))
\[
(1) \quad x^n > [x]^n + 1.
\]

(To see (1) use the binomial expansion for \( ([x] + r)^n \) and see the term \( \binom{n}{1} \cdot r \cdot [x]^{n-1} \).)

Finally, note that one can find an integer \( n > \frac{1}{2} \) such that \( [x^n] = [x]^n \) by our hypothesis about the infinitude of such numbers. Inequality (1) contradicts \( [x^n] = [x]^n \); so indeed \( x \) must be a positive integer.

**Problem 5.** Assume \( n \geq 2 \) and
\[
\frac{x^{n-1} - 1}{x - 1} \cdot \frac{x^n - 1}{x - 1} \cdot \frac{x^{n+1} - 1}{x - 1} \in \mathbb{Z}.
\]

So,
\[
x^{n-2} + \cdots + 1 \in \mathbb{Z}
\]
\[
x^{n-1} + \cdots + 1 \in \mathbb{Z}
\]
\[ x^n + \cdots + 1 \in \mathbb{Z}. \]

Subtracting the above expressions we obtain that \( x^n - 1, x^n \in \mathbb{Z} \) and thus \( x \in \mathbb{Q} \). If \( x \in \mathbb{Z} \), then we are done since then for each \( m \in \mathbb{N} \), we have

\[ \frac{x^n - 1}{x - 1} = \sum_{k=0}^{m-1} x^k \in \mathbb{Z}. \]

So, from now on, assume that \( x = \frac{a}{b} \), with \( a, b \in \mathbb{Z}, b \geq 2 \) and gcd\((a, b) = 1\). Then

\[ \frac{x^n - 1}{x - 1} = \frac{a^n - b^n}{b^{n-1}(a - b)}. \]

Since gcd\((a, b) = 1\) we immediately get that gcd\((a^n - b^n, b) = 1\); thus

\[ \frac{a^n - b^n}{b^{n-1}(a - b)} \notin \mathbb{Z} \]

(also note that \( n \geq 2 \)). In conclusion it must be that \( x \in \mathbb{Z} \), as desired.

Problem 6. We note one solution \( x = 1 \) and \( y = z = 0 \). Then we consider an arbitrary line in the 3-dimensional space passing through the point \((1, 0, 0)\). The equation of the line is

\[ \frac{x - 1}{t} = \frac{y}{u} = \frac{z}{v}. \]

where \( t, u, v \) are arbitrary. We let

\[ s := \frac{x - 1}{t} = \frac{y}{u} = \frac{z}{v}. \]

We search for \( s \) such that a point \((x, y, z)\) on the above line is also a solution to

\[ x^2 + y^2 + z^2 = 1. \]

Then using that \( x = st + 1, y = su \) and \( z = sv \) we obtain

\[ s^2t^2 + 2st + 1 + s^2u^2 + s^2v^2 = 1, \]

and so,

\[ s = -\frac{2t}{t^2 + u^2 + v^2}. \]

We solve for \( x, y, z \) and obtain

\[ x = -\frac{t^2 + u^2 + v^2}{t^2 + u^2 + v^2}, \]

\[ y = -\frac{2tu}{2tv}, \]

\[ z = -\frac{2tv}{2tv}. \]

Hence, varying \( t, u, v \in \mathbb{Q} \) (not equal to 0 and also satisfying \( u^2 + v^2 - t^2 \neq 0 \)), we obtain infinitely many nonzero solutions in rational numbers to the equation

\[ x^2 + y^2 + z^2 = 1. \]

Problem 7. If \( p = 2 \), then

\[ 3^2 + 19 \cdot (2 - 1) = 28 \text{ is not a perfect square.} \]

If \( p = 3 \), then

\[ 3^3 + 19 \cdot (3 - 1) = 65 \text{ is not a perfect square.} \]
From now on assume $p \geq 5$. We know that (by Fermat’s Little Theorem)
\[ 3^p \equiv 3 \pmod{p} \]
and so,
\[ 3^p + 19(p-1) \equiv -16 \pmod{p}. \]
So, if there exists an integer $x$ such that $x^2 = 3^p + 19(p-1)$ then
\[ x^2 \equiv -16 \pmod{p}. \]
Since $p$ is odd, we let $y$ be the inverse of 4 modulo $p$ then
\[ 4y \equiv 1 \pmod{p}, \]
and so,
\[ (xy)^2 \equiv -16y^2 \equiv -(4y)^2 \equiv -1 \pmod{p}. \]
So, $-1$ is a perfect square modulo $p$ which yields that $p \equiv 1 \pmod{4}$.

Now we consider modulo 4 (using that $p$ is odd)
\[ 3^p + 19(p-1) \equiv (-1)^p = -1 \pmod{4}. \]
However, $-1$ is not a perfect square modulo 4 which shows that indeed for all prime numbers $p$ we have
\[ 3^p + 19(p-1) \text{ is not a perfect square.} \]

**Problem 8.**

(i) We let $z = t^2$ and then
\[ 2z^3 + 8 = 2t^6 + 8 = (t^3 + 2)^2 + (t^3 - 2)^2. \]

Letting $t$ be an arbitrary integer yields infinitely many solutions.

(ii) We let $c = 3k + 1$ and then
\[ c^2 + 3 = 9k^2 + 6k + 4 = (3k - 2)^2 + 18k. \]

So, letting $k = 18\ell^2$ for each $\ell \in \mathbb{Z}$ we have the solution
\[ a = 54\ell^2 - 2; b = 18\ell; c = 54\ell^2 + 1. \]

**Problem 9.** We have
\[ x^2 + (y - 4z)^2 = (7z)^2. \]
Since $x$ is prime, we have that either $\gcd(x, y - 4z) = 1$ or $x \mid y - 4z$. In the latter case we would have that also $x \mid 7z$. So, letting $y - 4z = x \cdot y_1$ and $7z = x \cdot z_1$ for some integers $y_1$ and $z_1$ then we obtain
\[ 1^2 + y_1^2 = z_1^2. \]
The only solution is $z_1 = 1$ (note that $z > 0$ and so, $z_1 > 0$) and $y_1 = 0$. So, $7z = x$ and since $x$ is prime we conclude that $z = 1$ and $x = 7$. Also from $y_1 = 0$ we get $y - 4z = 0$ and therefore $y = 4$.

Now consider the case $\gcd(x, y - 4z) = 1$. Then $x$, $|y - 4z|$ and $7z$ are positive (since if $y - 4z$ were 0 then it would be divisible by $x$) integers relatively prime which are also a solution to the diophantine equation
\[ A^2 + B^2 = C^2. \]
Now, if $x$ is even, then $x = 2$ (since it’s prime) but then we obtain a contradiction since

$$4 + B^2 = C^2$$

has no solution in positive integers. So, from now on assume $x$ is odd. Then we know there exist positive integers $a$ and $b$ relatively prime such that

$$x = a^2 - b^2; \quad |y - 4z| = 2ab; \quad 7z = a^2 + b^2.$$  

But then $7 \mid a^2 + b^2$ which yields (since $7$ is a prime congruent with $3$ modulo $4$) that $7 \mid a$ and $7 \mid b$. This contradicts the fact that $a$ and $b$ are relatively prime.

In conclusion the only solution is $(x, y, z) = (7, 4, 1)$.

**Problem 10. Case 1.** $p \nmid a$

Then $\gcd(a, a + p) = \gcd(a, p) = 1$ and so,$$
a(a + p) = b^2$$
yields that there exist $a_1, a_2 \in \mathbb{N}$ such that $a = a_1^2$ and $a + p = a_2^2$. Then

$$p = a_2^2 - a_1^2 = (a_2 - a_1)(a_2 + a_1).$$

Because $p$ is prime we conclude that $a_2 - a_1 = 1$ and $a_2 + a_1 = p$. So, $a_2 = (p + 1)/2$ and $a_1 = (p - 1)/2$. Therefore,

$$a = \frac{(p - 1)^2}{4} \text{ and } b = \frac{p^2 - 1}{4}.$$  

**Case 2.** $p \mid a$

Then $a = pa_3$ with $a_3 \in \mathbb{N}$ and so,

$$p^2a_3(a_3 + 1) = b^2.$$  

Therefore $b = pb_1$ with $b_1 \in \mathbb{N}$ and so,

$$a_3(a_3 + 1) = b_1^2.$$  

Since $\gcd(a_3, a_3 + 1) = 1$ we conclude that both $a_3$ and $a_3 + 1$ are perfect squares, which is impossible (assuming both are positive integers).

So, the solution is

$$a = \frac{(p - 1)^2}{4} \text{ and } b = \frac{p^2 - 1}{4}.$$  

**Problem 11.**

(a) Assume that $\Delta := b^2 - 4ac \neq 0$.

Then, noting that the derivative $f'(x) = 2ax + b$ we obtain

$$4a \cdot f(x) - (f'(x))^2 = 4ac - b^2 = -\Delta \neq 0.$$  

Therefore, the only prime numbers $p$ dividing both $f(n)$ and $f'(n)$ for some $n \in \mathbb{Z}$ are the primes dividing $\Delta$, which yields a finite set of primes $p$ dividing both $f(n)$ and $f'(n)$.

(b) If $c = 0$, then the statement is obvious since for each prime $p$ we would have $p \mid ap^2 + bp = f(p)$. So, from now on assume $c \neq 0$.

Assume there exist finitely many such primes: $p_1, \ldots, p_k$ which divide some $f(n)$ for $n \in \mathbb{Z}$. We let

$$N = \ell \cdot c \cdot \prod_{i=1}^{k} p_i,$$
We know that

Problem 12. We know that

\[ r_1 + r_2 + r_3 = 0 \]
But also, we know that
\[ r_1 r_2 + r_1 r_3 + r_2 r_3 = a \]
and thus
\[ -3r_1^2 = a, \]
which proves that \( r_1 \in \mathbb{Q} \). Finally, we know that
\[ r_1 r_2 = -b \]
and so, \( -2r_1^3 = -b \), thus proving that \( r_1 \in \mathbb{Q} \). Finally, we know that
\[ r_1 r_2 r_3 = -c \]
and so, \( -r_3 r_1^2 = -c \), thus proving that \( r_3 \in \mathbb{Q} \). In conclusion, \( r_1, r_2, r_3 \in \mathbb{Q} \) and so, also \( r_3 \in \mathbb{Q} \). However, any rational solution to a monic equation with integer coefficients must be an integer, i.e., if \( \frac{a}{b} \) (in lowest terms) is a root of the monic polynomial equation
\[ x^d + c_{d-1} x^{d-1} + \cdots + c_1 x + c_0 = 0, \]
where each \( c_i \) is an integer, then we would get that
\[ a^d = - \sum_{i=0}^{d-1} c_i b^{d-i} a^i \]
and so, if there exists a prime \( p \) dividing \( b \), then we would have that also \( p \mid a \), contradicting the fact that \( \frac{a}{b} \) is in lowest terms. Therefore, indeed, any rational root of a monic polynomial with integer coefficients must be an integer itself, thus proving that \( r_1, r_2, r_3 \in \mathbb{Z} \).

**Problem 13.** By Problem 12, we know that \( r_1 = r_2 = r \) and \( r_3 = s \) must be integers. So,
\[ x^3 + ax + b = (x - r)^2 \cdot (x - s). \]
Therefore, for any positive integer \( n \) such that \( n - s \) is not a perfect square, we also have that \( f(n) \) is not a perfect square.

**Problem 14.** We let
\[ f(x) = \sum_{i=0}^{d} c_i x^i \]
and then expand
\[ f(n_0 + \ell p) \]
\[ = f(n_0) + \ell p \cdot f'(n_0) + \sum_{i=2}^{d} c_i \cdot \sum_{j=2}^{i} \binom{i}{j} n_0^{i-j} \]
\[ \equiv f(n_0) + \ell p f'(n_0) \pmod{p^2}. \]
So, if \( p^2 \mid f(n_0) \) and also \( p^2 \mid f(n_0 + \ell p) \), then we must have that \( p \mid f'(n_0) \) since \( p \nmid \ell \).

**Problem 15.** We apply the Euclidean Algorithm in \( \mathbb{Q}[x] \), i.e., divide repeatedly the polynomials, recording the remainder each time. More precisely, assuming (without loss of generality) that \( \deg(f) \geq \deg(g) \), we write
\[ f(x) = g(x) \cdot Q_1(x) + R_1(x) \text{ with } \deg(R_1) < \deg(g). \]
Then we write
\[ g(x) = R_1(x) \cdot Q_2(x) + R_2(x) \text{ with } \deg(R_2) < \deg(R_1) \]
and then
\[ R_1(x) = R_2(x) \cdot Q_3(x) + R_3(x) \text{ with } \deg(R_3) < \deg(R_2). \]
We continue until we are left with a remainder $R_m(x)$ which is a constant polynomial (since the degrees of the polynomials $R_i(x)$ decrease with $i$).

We claim that the first time when this happens (i.e., for the minimal such $m$) we have that $R_m(x) \neq 0$. Indeed, if $R_m(x) = 0$, then we would have that the polynomial $R_{m-1}(x)$ must divide $R_{m-2}(x)$ and furthermore (since $m$ is minimal), $R_{m-1}(x)$ is a nonconstant polynomial. Hence, there exists a root $r$ of $R_{m-1}(x) = 0$ and clearly (since $R_{m-2}(x) = Q_m(x) \cdot R_{m-1}(x)$) we have that also $R_{m-1}$ has the root $r$. Arguing backwards, starting with the polynomials $R_{m-1}$ and $R_{m-2}$, which share the same root $r$, we obtain that also $R_{m-3}$ and in general, each $R_i(x)$ has the root $r$. In particular, $g(x)$ and $R_1(x)$ would share the root $r$, which would then force also $f(x)$ to have the root $r$, contradicting the fact that $f(x)$ and $g(x)$ share no common root. Therefore, indeed, the first time when $R_m(x)$ is a constant polynomial, we have that $R_m(x) \neq 0$.

We let $c := R_m(x)$ (which is a constant in $\mathbb{Q}$ since all the polynomials $Q_i(x)$ and $R_i(x)$ have rational coefficients) and then note that

$$c = 1 \cdot R_{m-1}(x) - Q_m(x) \cdot R_{m-2}(x).$$

We continue solving backwards for the $R_i(x)$ (exactly as we do in the usual Euclidean Algorithm when dealing with the greatest common divisor of integers) and therefore, we derive the existence of some polynomials $P_0, Q_0 \in \mathbb{Q}[x]$ such that

$$c = f(x)P_0(x) + g(x)Q_0(x).$$

Finally, multiplying by some positive integer $N$ to clear all the possible denominators in $c$ and in each coefficient of $P_0(x)$ and of $Q_0(x)$, we obtain the existence of a nonzero integer $M$ and also, the existence of two polynomials $P$ and $Q$ with integer coefficients such that

$$f(x)P(x) + g(x)Q(x) = M.$$

**Problem 16.** Let $f(x) = \sum_{i=0}^{d} c_i x^i$. Clearly, if $c_0 = 0$, then for each prime $p$ and for each $k \in \mathbb{N}$, we have that $f(pk)$ is divisible by $p$ and therefore, each prime $p$ is in the set $S$. So, from now on, we assume $c_1 \neq 0$.

Assume the set $S$ is finite. We let $S_1$ be the set of primes appearing in $S$ along with the finitely many primes which may divide $c_0$ (if $c_0 = \pm 1$ then $S_1 = S$). We let

$$P := \prod_{p \in S_1} p.$$

If $S_1$ were empty, then we simply let $P = 1$.

Let $N$ be a sufficiently large positive integer such that $|f(x)| > 1 + |c_0|$ for all $x \geq N$. Then

$$f(NP|c_0|) = |c_0| \cdot \left( \epsilon_0 + P \cdot \sum_{i=1}^{d} c_i (P|c_0|)^{i-1} N^i \right),$$

where $\epsilon_0 := \frac{2c_0}{|c_0|} \in \{-1, 1\}$. Since $f(NP|c_0|) > 1 + |c_0|$ according to our hypothesis, we must have that the integer

$$N_1 := \epsilon_0 + P \cdot \sum_{i=1}^{d} c_i (P|c_0|)^{i-1} N^i$$
is an integer not equal to 0, 1 or −1. So, there exists a prime \( q \) dividing \( N_1 \); furthermore this prime \( q \) does not divide \( P \) and so, it cannot be in the set \( S \). However, once we know that \( q \) divides \( N_1 \) and thus, \( q \mid f(NP[c_0]) \) we must also have that \( q \) divides \( f(NP[c_0] + q \ell) \) for any positive integer \( \ell \) (note that \( P(a) \equiv P(b) \) \((\text{mod} \ q) \) if \( a \equiv b \) \((\text{mod} \ q) \)); so, \( q \) must belong to the set \( S \), contradiction. Therefore, indeed, \( S \) must be an infinite set.

**Problem 17.** Assume the contrary and therefore, assume that there exists some positive integer \( N_0 \) such that for all \( n \geq N_0 \), we have that \( f(n) \) is a perfect square.

Since all the (complex) roots of \( f(x) \) are distinct, then \( f(x) \) and its derivative \( f'(x) \) share no common root. Then (since they are both polynomials with integer coefficients), **Problem 4** yields the existence of a nonzero integer \( m \) and also, the existence of polynomials \( P, Q \in \mathbb{Z}[x] \) such that

\[
 f(x)P(x) + f'(x)Q(x) = m. 
\]

Now, let \( S \) be the set from **Problem 5** associated to \( f(x) \), i.e., the set of all primes dividing \( f(n) \) for (infinitely many) positive integers \( n \). Since we know now that \( S \) is infinite, we can choose a prime \( q \in S \) which does not divide the nonzero integer \( m \).

Now, let \( n_0 \geq N_0 \) be an integer such that \( q \mid f(n_0) \); in particular, because (by our assumption) \( f(n_0) \) is a perfect square, then we must have that \( q^2 \mid f(n_0) \). Now, for each \( \ell \in \mathbb{N} \), we have that

\[
 f(n_0 + \ell q) \equiv f(n_0) \equiv 0 \pmod{q} 
\]

and so, \( q \mid f(n_0 + \ell q) \) and since \( n_0 + \ell q > N_0 \), we get that also \( f(n_0 + \ell q) \) is a perfect square, which means that \( q^2 \mid f(n_0 + \ell q) \).

Using the conclusion of **Problem 3** for \( \ell \) not divisible by \( q \), we obtain that \( q \mid f'(n_0) \). But then \( q \) divides \( f(n_0) \cdot P(n_0) + f'(n_0) \cdot Q(n_0) = m \), which is a contradiction. So, in conclusion, there must be infinitely many positive integers \( n \) such that \( f(n) \) is not a perfect square.

**Problem 18.** We deal with a cubic polynomial with integer coefficients:

\[
 f(x) = ax^3 + bx^2 + cx + d, \text{ where } a \neq 0. 
\]

If \( a < 0 \) then \( \lim_{n \to \infty} f(n) = -\infty \) and so, clearly, there exist infinitely many positive integers \( n \) such that \( f(n) \) is not a perfect square. So, from now on, we assume \( a > 0 \).

By **Problem 17**, we know that if all roots of \( f(x) \) are distinct, then we are done. So, from now on, we assume there exists a double root \( r \) for \( f(x) \). Immediately this proves that all roots of \( f(x) \) are rational (with the proof essentially as for **Problem 12**). Indeed, we argue using linear transformations to modify the above cubic polynomial into a cubic polynomial of the form \( a^3 + Ax + B \). In order to do this, we note that solving \( f(x) = 0 \) is equivalent with solving \( 27a^2 f(x) = 0 \), i.e.,

\[
 (3ax)^3 + 3b(3ax)^2 + 9ca(3ax) + 27da^2 = 0
\]

and so, it suffices to prove that the roots of

\[
 x^3 + 3bx^2 + 9cax + 27da^2 = 0
\]

are rational. This yields the equation:

\[
 (x + b)^3 + (9ca - 3b^2)(x + b) + 27da^2 + 2b^3 - 9abc = 0,
\]
and therefore, the roots of \( f(x) = 0 \) are rational if and only if the roots of 
\[
x^3 + Ax + B = 0
\]
are rational, where \( A = 9ca - 3b^2 \) and \( B = 27da^2 + 2b^3 - 9abc \); furthermore, since we can navigate between the two equations through the above linear transformations, once we know that the original equation has a double root, then also the equation 
\[
x^3 + Ax + B = 0
\]
has a double root and therefore, by Problem 12, all the roots of \( f(x) = 0 \) are rational (we can no longer guarantee that they are integral since the above linear transformations performed in reverse order may introduce denominators even though we know we start with some integral solutions to the equation \( x^3 + Ax + B = 0 \)).

So, we know that \( f(x) = a(x - r)^2(x - s) \). We write \( s := \frac{N}{D} \) for some coprime integers \( D \). Pick a prime \( p > \max\{a, |N|, D\} \) and then let \( n_0 \in \mathbb{N} \) such that 
\[
n_0D \equiv N \pmod{p},
\]
which is possible since \( D \) is invertible modulo \( p \). Then for each \( \ell \in \mathbb{N} \), the exponent of the prime \( p \) in \( f(n_0 + \ell p) \) has the same parity as the exponent of the prime \( p \) in \( nD - N \) and therefore, since
\[
p \mid (n_0 + \ell p)D - N,
\]
then for each prime \( \ell \) such that \( f(n_0 + \ell p) \) is a perfect square, we must have that
\[
\exp_p ((n_0 + \ell p)D - N) \geq 2.
\]
But the difference between any two consecutive integers in the arithmetic progressions \( \{(n_0 + \ell p)D - N\}_{\ell \in \mathbb{N}} \) is \( pD \), while the difference between multiples of \( p^2 \) is at least \( p^2 \), contradiction. So, indeed, there must be infinitely many \( n \) such that \( f(n) \) is not a perfect square.

**Problem 19.** Let \( n \) be a positive integer such that \( f(n) = m^2 \) for some nonnegative integer \( m \). So, we have
\[
m^2 = n^3 + 27
\]
and analyzing this diophantine equation modulo 4, we get two cases.

**Case 1.** \( n \) is even.

In this Case 1, we have that \( n^3 \equiv 0 \pmod{8} \) and so,
\[
m^2 \equiv n^3 + 27 \equiv 3 \pmod{8},
\]
which is impossible. Therefore, we must have only

**Case 2.** \( n \) is odd.

In this Case 2, we get that \( m^2 \) and therefore \( m \) itself must be even (since \( n^3 + 27 \) is even in this Case 2). So,
\[
n^3 + 27 \equiv m^2 \equiv 0 \pmod{4},
\]
which yields that \( n^3 \equiv 1 \pmod{4} \). Since already we have that \( n^2 \equiv 1 \pmod{4} \) (because \( n \) is odd) then we must have that \( n \) itself is 1 modulo 4.

We also have
\[
m^2 + 4 = n^3 + 27 = (n + 3)(n^2 - 3n + 9)
\]
and since $n \equiv -1 \pmod{4}$, we get that

$$n^2 - 3n + 9 \equiv -1 \pmod{4}.$$ 

On the other hand, for any positive integer $n$, we have that

$$n^2 - 3n + 9 = \left( n - \frac{3}{2} \right)^2 + \frac{27}{4} > 6.$$ 

So, the number $n^2 - 3n + 9$ is greater than 6 and it is $-1$ modulo 4; therefore, it must be divisible by some prime $q$ which is itself $-1$ modulo 4. But then this prime number $q$ must also divide $n^2 + 4$, which is impossible since this would mean that $-4$ (and therefore $-1$) is a quadratic residue modulo $q$ (which contradicts the fact that $q \equiv -1 \pmod{4}$). In conclusion, there exists no positive integer $n$ such that $f(n)$ is a perfect square.