1. Problems

**Problem 1.** A particle moves on the \(x\)-axis and its position at time \(t\) (measured in seconds) is given by the function \(s(t) = t^3 - 27t + 1\) (measured in meters).

(i) What is the average velocity of the particle in the first 4 seconds?

(ii) What is the total distance traveled by the particle in the first 4 seconds?

**Problem 2.** Find all points on the curve \(y = x^2\) where the tangent line to the given curve at that point passes through the point \((0, -5)\).

**Problem 3.** Find \(a\) and \(b\) such that the tangent line at the point \((1, 5)\) on the graph of \(y = ax^3 + bx^2\) has equation \(y = 7x - 2\).

**Problem 4.** If \(f(x) = \sin(x) \cdot g(x)\) and \(g(\pi) = 2\) and \(g'(\pi) = -3\), find \(f'(\pi)\).

2. Solutions

**Problem 1.**

(i) The average velocity in the first 4 seconds is

\[
v_{\text{average}} = \frac{s(4) - s(0)}{4 - 0}.
\]

We compute \(s(4) = 4^3 - 4 \cdot 27 + 1 = 64 - 108 + 1 = -43\) and \(s(0) = 1\). So,

\[
v_{\text{average}} = \frac{-43 - 1}{4} = -11 \text{ m/s}.
\]

(ii) Already as shown by the computation from (i), we see that the particle moved (partially) backwards in the first 4 seconds. To determine the period when the particle moves backwards, we compute the velocity and see when it is negative. The velocity is the derivative of the position function, i.e.

\[
v(t) = s'(t) = 3t^2 - 27.
\]

So, \(v(t) < 0\) when \(3t^2 - 27 < 0\), i.e. when \(t^2 < 9\). Thus, the particle had negative velocity, i.e. it moved backwards in the time interval 0 to 3 seconds.

Initially, when \(t = 0\), the particle was situated at position \(s(0) = 1\) on the \(x\)-axis. Then at \(t = 3\), the particle is situated at \(s(3) = 3^3 - 27 \cdot 3 + 1 = -53\). So, in the time interval 0 to 3 seconds, the particle traveled \(1 - (-53) = 54\) meters.

Then in the time interval 3 to 4 seconds, the particle moves forward (since the velocity \(v(t) > 0\) for \(t > 3\)). The position of the particle at \(t = 4\)
is \( s(4) = -43 \). So, in the time interval 3 to 4 seconds, the particle traveled 
\[ -43 - (-53) = 10 \text{ meters}. \]
Hence, in total, in the time interval 0 to 4 seconds, the particle traveled 
\[ 54 + 10 = 64 \text{ meters}. \]

**Problem 2.** Let \((a, a^2)\) be a point on the curve \( y = x^2 \) where the tangent line to 
the given curve passes also through the point \((0, -5)\). Note that the point \((0, -5)\) is 
not on the parabola \( y = x^2 \); it lies below its graph. However, the tangent line at 
the point \((a, a^2)\) (which on the other hand is on the parabola) is assumed to pass 
also through the point \((0, -5)\). Since this tangent line is assumed to pass through 
the two points: \((a, a^2)\) and \((0, -5)\), we can compute its slope as 
\[ \frac{\Delta y}{\Delta x} = \frac{-5 - a^2}{0 - a} = \frac{-5 - a^2}{-a} = \frac{a^2 + 5}{a}. \]
On the other hand, the slope of the tangent line at the curve \( y = x^2 \) computed at 
the point with \( x \)-coordinate \( a \) is simply the derivative of \( x^2 \) computed at \( x = a \), 
i.e., the slope of this tangent line is equal to \( 2a \). So, we equate the two formulas we 
obtained for the slope of the tangent line passing through the point \((0, -5)\) and get 
\[ \frac{a^2 + 5}{a} = 2a, \]
which yields \( a^2 + 5 = 2a^2 \), i.e. \( a^2 = 5 \). We conclude that \( a = \sqrt{5} \) or \( a = -\sqrt{5} \). In 
conclusion, the points on the parabola \( y = x^2 \) with the property that the tangent 
lines constructed at these points also pass through the point \((0, -5)\) are \((\sqrt{5}, 5)\) and 
\((-\sqrt{5}, 5)\).

**Problem 3.** We are given that the point \((1, 5)\) is on the graph of \( y = ax^3 + bx \), 
i.e. plugging in \( x = 1 \) and \( y = 5 \) verifies the above equation, which yields 
\[ 5 = a \cdot 1^3 + b \cdot 1, \]
i.e. \( 5 = a + b \). On the other hand, we are told that the tangent line at this point has 
equation \( y = 7x - 2 \), which means that it has slope equal to 7. However, the slope of 
the tangent at the point with \( x \)-coordinate equal to 1 on the graph of \( y = ax^3 + bx \) 
is the derivative of \( ax^3 + bx \) evaluated at \( x = 1 \). Since the derivative of \( ax^3 + bx \) is 
\( 3ax^2 + b \), we get that 
\[ 3a + b = 7. \]
So, we have a system of two equations with the unknowns:
\[
\begin{align*}
    a + b &= 5 \\
    3a + b &= 7
\end{align*}
\]
We solve and obtain \( a = 1 \) and \( b = 4 \).

**Problem 4.** We compute using Product Rule: 
\[ f'(x) = \cos(x) \cdot g(x) + \sin(x) \cdot g'(x). \]
Then using that \( g(\pi) = 2 \) while \( g'(\pi) = -3 \), we conclude that 
\[ f'(\pi) = \cos(\pi)g(\pi) + \sin(\pi) + g'(\pi) = (-1) \cdot 2 + 0 \cdot (-3) = -2. \]