1. Problems

Problem 1. Find the following limits

(a) \( \lim_{x \to +\infty} x^{\frac{1}{x}} \)

(b) \( \lim_{x \to 0^+} x \cdot e^{\frac{1}{x}} \)

(c) \( \lim_{x \to 0^+} x^2 \cdot \ln(x) \)

Problem 2. Let \( f(x) = 9x^{\frac{2}{7}} - 2x^{\frac{1}{7}} \).

(a) Determine the intervals where \( f(x) \) is increasing, and the intervals where \( f(x) \) is decreasing.

(b) Find the \( x \) and \( y \) coordinates of the points of local maximum and of the points of local minimum for \( f(x) \).

(c) Determine the intervals where \( f(x) \) is concave up, and the intervals where \( f(x) \) is concave down.

(d) Find the \( x \)-coordinates of the inflection points for \( f(x) \).

2. Solutions

Problem 1.

(a) We let

\[ L := \lim_{x \to +\infty} x^{\frac{1}{x}}. \]

Then

\[ \ln(L) = \lim_{x \to +\infty} \frac{\ln(x)}{x}. \]

We compute the above limit using L'Hôpital’s Rule since it’s a limit of type \( \frac{+\infty}{+\infty} \):

\[ \lim_{x \to +\infty} \frac{\ln(x)}{x} = \lim_{x \to +\infty} \frac{\frac{1}{x}}{1} = 0. \]

So, \( \ln(L) = 0 \) and thus the limit \( L = 1 \).
(b) We turn the product into a quotient so we can apply the L'Hôpital's Rule:

\[
\lim_{x \to 0^+} x \cdot e^{\frac{1}{x}} = \lim_{x \to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}}.
\]

The last limit is of type \( \frac{\pm \infty}{\pm \infty} \) and so,

\[
\lim_{x \to 0^+} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = \lim_{x \to 0^+} e^{\frac{1}{x}} \cdot \left( -\frac{1}{x^2} \right) = \lim_{x \to 0^+} e^{\frac{1}{x}},
\]

which diverges to \(+\infty\).

(c) Again we turn a product into a quotient so we can apply L'Hôpital's Rule:

\[
\lim_{x \to 0^+} x^2 \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x^2}}.
\]

The last limit is of type \( \frac{-\infty}{\mp \infty} \) and so,

\[
\lim_{x \to 0^+} \frac{\ln(x)}{\frac{1}{x^2}} = \lim_{x \to 0^+} -\frac{2}{x} = \lim_{x \to 0^+} -\frac{2}{2} = 0.
\]

**Problem 2.**

(a) We differentiate

\[
f'(x) = \frac{18}{7} \cdot x^{-\frac{5}{7}} - \frac{18}{7} \cdot x^{\frac{2}{7}}
\]

\[
= \frac{18}{7x^{\frac{5}{7}}} - \frac{18}{7} \cdot x^{\frac{2}{7}}
\]

\[
= \frac{18}{7x^{\frac{5}{7}}} - \frac{18}{7} \cdot x^{\frac{2}{7}}
\]

\[
= \frac{18}{7} \left( 1 - x \right).
\]

So, the sign of \( f'(x) \) is determined by both \( 1 - x \) and \( x \); \( f'(x) > 0 \) when both are positive (it can’t be that both \( x \) and \( 1 - x \) are negative since their sum is \( 1 > 0 \)). In conclusion, \( f'(x) > 0 \) if \( x \in (0, 1) \). So, \( f(x) \) is decreasing on \((-\infty, 0) \cup (1, +\infty)\) and increasing on \((0, 1)\).

(b) Using the information from (a), we conclude that \( f(x) \) has a local minimum at \( x = 0 \), which equals \( f(0) = 0 \) and a local maximum at \( x = 1 \) which equals \( f(1) = 7 \).

(c) We differentiate again

\[
f''(x) = -\frac{90}{49} \cdot x^{-\frac{12}{7}} - \frac{36}{49} \cdot x^{-\frac{5}{7}}
\]

\[
= -\frac{90}{49} \cdot x^{-\frac{12}{7}} - \frac{36}{49} \cdot x^{-\frac{5}{7}}
\]

\[
= -\frac{90}{49} \cdot x^{-\frac{12}{7}} + \frac{36}{49} \cdot x^{-\frac{5}{7}}
\]

\[
= -\frac{90}{49} \cdot x^{-\frac{12}{7}} - \frac{36}{49} \cdot x^{-\frac{5}{7}}
\]

\[
= -\frac{18}{49} \cdot (5 + 2x)
\]

\[
= -\frac{18}{49} \cdot \frac{5 + 2x}{x^{\frac{12}{7}}}.
\]
So, the sign is determined this time only by the numerator since the denominator is positive for all (nonzero) $x$ due to the power $\frac{12}{7}$. So, we conclude that $f''(x) > 0$ if $5 + 2x$ is negative, i.e. for $x < \frac{-5}{2}$. In conclusion, $f(x)$ is concave up on $(-\infty, \frac{-5}{2})$ and it is concave down on $(\frac{-5}{2}, 0) \cup (0, +\infty)$ (we separated into two intervals since at $x = 0$, $f''(x)$ and even $f'(x)$ is not defined).

(d) The only inflection point is at $x = \frac{-5}{2}$. 