1. Problems

Problem 1. Find arcsin \( \sin \left( \frac{11\pi}{3} \right) \).

Problem 2. Find the derivative of the functions:
   (a) \( y(x) = 5^x \)
   (b) \( y(x) = \sin(x) \ln(x) \)

Problem 3. Find the inverse function for \( f(x) = 5^x \).

Problem 4. Find the inverse function for \( f(x) = 3^x^2 \) where \( f : [0, +\infty) \rightarrow [1, +\infty) \).

Problem 5.
   (1) If \( f(x) = e^x + x^e \) for positive real numbers \( x \), and \( g(x) \) is the inverse function for \( f(x) \), find \( g(2e^e) \).
   (2) If \( f(x) = (x^2 + 1)^x \) for positive real numbers \( x \), and \( g(x) \) is the inverse of \( f(x) \), find \( g(2) \).

2. Solutions.

Problem 1. \( \arcsin \left( \sin \left( \frac{11\pi}{3} \right) \right) \) is an angle \( \theta \) with the properties:
   • \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\); and
   • \( \sin(\theta) = \sin \left( \frac{11\pi}{3} \right) \).

We know that \( \sin(x) \) is a function periodic of period \( 2\pi \); so,
\[
\sin \left( \frac{11\pi}{3} \right) = \sin \left( \frac{11\pi}{3} - 2\pi \right) = \sin \left( \frac{11\pi}{3} - 4\pi \right) = \sin \left( \frac{-\pi}{3} \right).
\]
So, \( \theta = -\frac{\pi}{3} \). Note that \( \frac{11\pi}{3} \) or \( \frac{11\pi}{3} - 2\pi = \frac{5\pi}{3} \) are not the correct answer since they are not in the interval \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \).

Problem 2.
(a) We can either differentiate directly and get
\[ y'(x) = 5^x \cdot \ln(5) \cdot (3^x)' = 5^x \cdot \ln(5) \cdot 3^x \cdot \ln(3). \]
Alternatively we can use logarithmic differentiation and get
\[ \ln(y(x)) = 3^x \cdot \ln(5) \]
and then we differentiate (note the chain rule for the left hand side)
\[ \frac{y'(x)}{y(x)} = 3^x \cdot \ln(3) \cdot \ln(5), \]
and so
\[ y'(x) = y(x) \cdot 3^x \cdot \ln(3) \cdot \ln(5) = 5^x \cdot 3^x \cdot \ln(3) \cdot \ln(5). \]
(b) We use logarithmic differentiation:
\[ \ln(y(x)) = \ln(x) \cdot \ln(\sin(x)), \]
and then differentiate
\[ \frac{y'(x)}{y(x)} = \frac{1}{x} \cdot \ln(\sin(x)) + \ln(x) \cdot \frac{\cos(x)}{\sin(x)}, \]
and so
\[ y'(x) = \sin(x)^{\ln(x)} \cdot \left( \frac{\ln(\sin(x))}{x} + \frac{\ln(x) \cdot \cos(x)}{\sin(x)} \right). \]

**Problem 3.** We let \( y = g(x) \) be the inverse function of \( f(x) \). Then
\[ f(g(x)) = x, \]
or in other words, \( f(y) = x \). But \( f(y) = 5^y \); so
\[ 5^y = x. \]
We apply the logarithm in base 5 to both sides and compute
\[ 3^y = \log_5(x). \]
Then we apply the logarithm in base 3 and get
\[ y = g(x) = \log_3(\log_5(x)) \]
is the inverse function for \( f(x) \).

**Second solution:** In order to find \( y = g(x) \) the inverse function of \( f(x) \), we could work only with the natural logarithm \( \ln \). So, from \( f(g(x)) = x \) which yields \( f(y) = x \) and thus
\[ 5^y = x. \]
We apply the natural logarithm and get
\[ 3^y \cdot \ln(5) = \ln(x), \]
and so,
\[ 3^y = \frac{\ln(x)}{\ln(5)}. \]
Then applying again the natural logarithm yields
\[ y \cdot \ln(3) = \ln \left( \frac{\ln(x)}{\ln(5)} \right), \]
and thus the inverse function is
\[ y = g(x) = \frac{\ln\left(\frac{\ln(x)}{\ln(5)}\right)}{\ln(3)}. \]

The two expressions for \( g(x) \) computed with the above two methods are the same since
\[ \log_5(x) = \frac{\ln(x)}{\ln(5)}. \]

Similarly, for any \( t \) we have
\[ \log_3(t) = \frac{\ln(t)}{\ln(3)}. \]

The above properties of the logarithm are just as important as the identities:
\[ \log_a(A \cdot B) = \log_a(A) + \log_a(B) \]
and
\[ \log_a(A^B) = B \cdot \log_a(A), \]
for any positive real numbers \( a, A \) and \( B \).

**Problem 4.** We have that \( y = 3x^2 \) and now we solve for \( x \) in terms of \( y \) in order to find the inverse function. So, we have (after taking logarithms of both sides)
\[ \log_3(y) = x^2. \]

Then we take square-roots and obtain
\[ x = \sqrt{\log_3(y)}. \]

So, the inverse function for \( f(x) \) is the function \( g : [1, +\infty) \rightarrow [0, +\infty) \) given by the formula
\[ f(x) = \sqrt{\log_3(x)}. \]

**Problem 5.** For these type of questions, we want to guess the value \( b = g(a) \) such that
\[ f(b) = f(g(a)) = a. \]

because \( g(x) \) is the inverse of \( f(x) \). We only guess the value \( b = g(a) \) since it is very hard to find the actual formula for \( g(x) \), the inverse function of \( f(x) \). But guessing the value \( b = g(a) \) turns out not to be very difficult.

(a) So, we have \( f(x) = e^x + x^2 \) and we want to find \( g(2e^e) \). So, we want to guess the value \( b = g(2e^e) \) such that \( f(g(2e^e)) = 2e^e \), i.e. \( f(b) = 2e^e \). In other words, \( b \) satisfies
\[ e^b + b^2 = 2e^e. \]

It’s not hard to guess that \( b = e \) works. So, \( g(2e^e) = e. \)

(b) This time we have \( f(x) = (1 + x^2)^e \) and we want to find \( g(2) \). So, letting \( b = g(2) \) then \( f(g(2)) = 2 \) and so, \( f(b) = 2 \). In other words, \( b \) satisfies
\[ (1 + b^2)^b = 2. \]

It’s not hard to guess that \( b = 1 \) works. So, \( g(2) = 1. \).