1. Problems

Problem 1. Determine the domain of continuity for the function

\[ f(x) = \begin{cases} 
\frac{x^2 - x - 2}{x - 1}, & \text{if } x < 2 \\
0, & \text{if } x = 2 \\
\sqrt{x + 1}, & \text{if } x > 2 
\end{cases} \]

Problem 2. For each of the following functions determine the corresponding domain of continuity.

(a) \( e^{\sin(x)} \)

(b) \( \sqrt{3 - x^2} \)

(c) \( \frac{\sqrt{1 + x}}{x - 2} \)

Problem 3. Determine the horizontal and the vertical asymptotes for the function \( \frac{\sqrt{2x^2 + 1}}{x - 1} \).

Problem 4. Evaluate the following limits:

(a) \( \lim_{x \to 4} \frac{\frac{1}{x} - \frac{1}{4}}{x - 4} \)

(b) \( \lim_{x \to -\infty} \arctan(e^x) \)

(c) \( \lim_{x \to +\infty} e^{-x} \sin(x) \)

(d) \( \lim_{x \to +\infty} \frac{x + 1}{\sqrt{x^3 - x + 1}} \)

(e) \( \lim_{x \to -\infty} \cos \left( \frac{1}{x} \right) \)

(f) \( \lim_{x \to +\infty} \sin(x) \)
(g) \[ \lim_{x \to -\infty} x^2 - 3x \]

(h) \[ \lim_{x \to +\infty} \sqrt{4x^2 + 5x + 7} - 2x \]

Problem 5. Show that the equation
\[ e^x - 2 \cos(x) = 0 \]
has at least one real solution.

Problem 6. Show that the equation
\[ x^3 - 7x - 4 = 0 \]
has three real roots.

Problem 7. Let
\[ f(x) = \begin{cases} \frac{1}{x-4} & \text{if } x \neq 4 \\ \frac{1}{8} & \text{if } x = 4 \end{cases} \]
and
\[ g(x) = \begin{cases} 2x & \text{if } x \leq 2 \\ x^2 & \text{if } x > 2 \end{cases} \]
Compute \[ \lim_{x \to 2} f(g(x)) \].

Problem 8. Find \( c \) such that the function
\[ f(x) = \begin{cases} cx + 1 & \text{if } x \leq 3 \\ cx^2 - 1 & \text{if } x > 3 \end{cases} \]
is continuous for all real values \( x \).

2. Solutions.

Problem 1. We first identify potential points of discontinuity on each branch. Note that there are 2 branches only: \( x < 2 \) and \( x > 2 \); \( x = 2 \) is just a point and never is counted as a branch by itself.

For \( x < 2 \), the only potential discontinuity appears when the denominator vanishes, i.e., \( x - 3 = 0 \), which yields \( x = 3 \). However, \( x = 3 \) is not part of this branch: \( x < 2 \); therefore \( f(x) \) is indeed continuous (as a well-defined quotient of continuous functions) for all \( x < 2 \).

For \( x > 2 \), in order for the function to be continuous (and well-defined!) we need: \( x + 1 \geq 0 \), i.e., \( x \geq -1 \). However, this condition is automatically satisfied on the branch for \( x > 2 \). Therefore \( f(x) \) is continuous for all \( x > 2 \).

Now, at \( x = 2 \), we always have to consider left and right limits and check that they are equal to \( f(2) \) since \( x = 2 \) is a branch point for \( f(x) \). So,
\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - x - 2}{x - 3} = \frac{2^2 - 2 - 2}{2 - 3} = 0 \]
and
\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \sqrt{x + 1} = \sqrt{3}.
\]
So, \(\lim_{x \to 2^-} f(x) = f(2) = 0\), but \(\lim_{x \to 2^+} f(x) \neq f(2)\), which yields that \(f(x)\) is not continuous at \(x = 2\). (The function is only continuous from the left at \(x = 2\).)

In conclusion, \(f(x)\) is continuous on \((-\infty, 2) \cup (2, +\infty)\).

**Problem 2.**

(a) Both \(\sin(x)\) and the exponential function are continuous for all values of \(x\); thus \(e^{\sin(x)}\) is continuous for all values of \(x\), i.e., the function is continuous on \(\mathbb{R}\).

(b) The function is continuous whenever it is well-defined. So, we need
\[
3 - x^2 \geq 0 \text{ i.e., } x^2 \leq 3.
\]
In conclusion, \(|x| \leq \sqrt{3}\) which yields \(-\sqrt{3} \leq x \leq \sqrt{3}\). So, the domain of continuity for the function is \([-\sqrt{3}, \sqrt{3}]\).

(c) The function is continuous whenever it is well-defined (since it is a quotient of an algebraic function by a polynomial). So, we need to enforce the conditions which make the function well-defined. Thus, we need
\[
x + 1 \geq 0 \text{ i.e., } x \geq -1
\]
so that the square root is well-defined, and we also need
\[
x - 2 \neq 0 \text{ i.e., } x \neq 2
\]
so that the denominator does not vanish. In conclusion, the function is continuous on \([−1, 2) \cup (2, +\infty)\).

**Problem 3.** We have a vertical asymptote at \(x = 1\) since
\[
\lim_{x \to 1^+} \frac{\sqrt{2x^2 + 1}}{x - 1} \text{ diverges to } +\infty.
\]
Indeed, the numerator converges to \(\sqrt{3}\) while the denominator converges to 0 (through positive values since \(x \to 1^+\)).

We also have two horizontal asymptotes: \(y = \sqrt{2}\) and \(y = -\sqrt{2}\) because
\[
\lim_{x \to +\infty} \frac{\sqrt{2x^2 + 1}}{x - 1} = \lim_{x \to +\infty} \frac{\sqrt{\frac{2x^2 + 1}{x^2}}}{\frac{x-1}{x}} = \lim_{x \to +\infty} \frac{\sqrt{\frac{2}{x^2} + \frac{1}{x^2}}}{\frac{1}{x} - \frac{1}{x}} = \sqrt{2}
\]
and
\[
\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{x - 1} = \lim_{x \to +\infty} \frac{\sqrt{2(-x)^2 + 1}}{-x - 1} = \lim_{x \to +\infty} -\frac{\sqrt{2x^2 + 1}}{x + 1} = -\lim_{x \to +\infty} \frac{\sqrt{2\frac{1}{x^2}}}{\frac{1}{x} + \frac{1}{x^2}} = -\lim_{x \to +\infty} \frac{\sqrt{2 + \frac{1}{x}}}{\frac{1}{x} + \frac{1}{x^2}} = -\sqrt{2}.
\]

**Problem 4.**

(a) We bring to a common denominator on top and obtain:
\[
\lim_{x \to 4} \frac{4 - x}{4x - 4} = \lim_{x \to 4} \frac{-(x - 4)}{4(x - 4)} = \lim_{x \to 4} \frac{-1}{4x} = -\frac{1}{16}.
\]

(b) As \( x \to -\infty, e^x \to 0 \). Since \( \arctan(0) = 0 \) (as \( \tan(0) = 0 \)) we conclude that
\[
\lim_{x \to -\infty} \arctan(e^x) = 0.
\]

(c) We know that

\[-1 \leq \sin(x) \leq 1.
\]

So,
\[
-\frac{1}{e^x} \leq e^{-x} \sin(x) \leq \frac{1}{e^x},
\]

and because
\[
\lim_{x \to +\infty} -\frac{1}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0,
\]

we conclude by Squeeze Theorem that also
\[
\lim_{x \to +\infty} e^{-x} \sin(x) = 0.
\]

(d) We divide by the highest power of \( x \) appearing in the denominator. Note that because of the fourth-root appearing in the denominator, the highest power of \( x \) from the denominator is \( x^{\frac{3}{4}} \). So,
\[
\lim_{x \to +\infty} \frac{x + 1}{\sqrt[4]{x^3} - x + 1} = \lim_{x \to +\infty} \frac{x^{\frac{1}{4}} + \frac{1}{x^{\frac{3}{4}}}}{\frac{1}{x^{\frac{3}{4}}} - \frac{1}{x^{\frac{1}{2}}} + \frac{1}{x}}
\]

diverges to \( +\infty \).

Indeed, note that all of the above fractions converge to 0; the only term which does not converge to 0, but rather diverges to \( +\infty \) is the first term of the numerator: \( x^{\frac{1}{4}} \).
(e) As \( x \to -\infty \) we have that \( \frac{1}{x} \) converges to 0, and since \( \cos(0) = 1 \), we conclude that 
\[
\lim_{x \to -\infty} \cos \left( \frac{1}{x} \right) = 1.
\]

(f) As \( x \to +\infty \), \( \sin(x) \) oscillates between \(-1\) and \(1\); therefore, 
\[
\lim_{x \to +\infty} \sin(x) \text{ does not exist.}
\]

(g) 
\[
\lim_{x \to -\infty} x^2 - 3x = \lim_{x \to -\infty} x(x - 3)
\]
As \( x \to -\infty \), we also have that \( x - 3 \to -\infty \), and therefore their product diverges to \(+\infty\). In conclusion, 
\[
\lim_{x \to -\infty} x^2 - 3x \text{ diverges to } +\infty.
\]

(h) \( \sqrt{4x^2 + 5x + 7} \) is of the order of \( \sqrt{4x^2} = 2x \) as \( x \) goes to \(+\infty\). However we cannot conclude that the limit is 0 (actually it’s not 0, as we will soon see!). The only way to deal with this limit is by multiplying (and dividing at the same time) with the conjugate:
\[
\sqrt{4x^2 + 5x + 7} = 2x.
\]

We obtain
\[
\lim_{x \to +\infty} \sqrt{4x^2 + 5x + 7} - 2x = \lim_{x \to +\infty} \frac{(\sqrt{4x^2 + 5x + 7} - 2x)(\sqrt{4x^2 + 5x + 7} + 2x)}{\sqrt{4x^2 + 5x + 7} + 2x} = \lim_{x \to +\infty} \frac{4x^2 + 5x + 7 - 4x^2}{\sqrt{4x^2 + 5x + 7} + 2x} = \lim_{x \to +\infty} \frac{5x + 7}{\sqrt{4x^2 + 5x + 7} + 2x} = \lim_{x \to +\infty} \frac{5 + \frac{7}{x}}{\sqrt{4 + \frac{5}{x} + \frac{7}{x^2} + \frac{2}{x^2}}} = \frac{5}{\sqrt{4 + 2}} = \frac{5}{4}.
\]

**Problem 5.** We let \( f(x) = e^x - 2 \cos(x) \) which is a continuous function since it’s a difference of two continuous functions. We evaluate:
\[
f(0) = e^0 - 2 \cdot \cos(0) = -1 < 0
\]
and
\[
f \left( \frac{\pi}{2} \right) = e^{\frac{\pi}{2}} - 2 \cos \left( \frac{\pi}{2} \right) = e^{\frac{\pi}{2}} > 0.
\]
Therefore by the Intermediate Value Theorem there exists \( c \in (0, \frac{\pi}{2}) \) such that \( f(c) = 0 \); so the equation 
\[
e^x - 2 \cos(x) = 0
\]
has at least one real root \( c \).

**Problem 6.** The polynomial \( f(x) = x^3 - 7x - 4 \) is continuous for all real numbers \( x \). We compute:
\[
f(-3) = -27 + 21 - 4 = -10 < 0
\]
\[ f(-1) = -1 + 7 - 4 = 2 > 0 \]
\[ f(0) = -4 < 0 \]
\[ f(3) = 27 - 21 - 4 = 2 > 0. \]

Therefore (because we have three changes of sign in the values of \( f(x) \)) by the Intermediate Value Theorem there exist \( c_1 \in (-3, -1), c_2 \in (-1, 0) \) and \( c_3 \in (0, 3) \) such that \( f(c_1) = f(c_2) = f(c_3) = 0 \). So, the above polynomial equation has three distinct roots; also it cannot have more than 3 roots since

\[ x^3 - 7x - 4 = 0 \]

is a cubic equation.

**Problem 7.** \( \lim_{x \to 2} g(x) \) exists (as we will see shortly); actually \( g(x) \) is continuous everywhere including at \( x = 2 \). Indeed,

\[
\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} 2x = 4
\]

and

\[
\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} x^2 = 4.
\]

So, \( \lim_{x \to 2} g(x) = 4 \) (which actually matches \( g(2) = 2 \cdot 2 = 4 \)).

However, \( f(x) \) is not continuous at \( x = 4 \) (\( f(x) \) has an infinite discontinuity at \( x = 4 \)) and therefore we cannot conclude that \( \lim_{x \to 2} f(g(x)) \) would equal \( f(\lim_{x \to 2} g(x)) = f(4) = 8 \). Instead, in order to compute \( \lim_{x \to 2} f(g(x)) \) we would have to consider left and right limits as \( x \) approaches 2. So,

\[
\lim_{x \to 2^-} f(g(x)) = \lim_{x \to 2^-} f(2x) = \lim_{x \to 2^-} \frac{1}{2x - 4}
\]

which diverges to \(-\infty\). **Already** this proves that

\[
\lim_{x \to 2} f(g(x)) \text{ does not exist.}
\]

But (for practice) we will also compute the right limit:

\[
\lim_{x \to 2^+} f(g(x)) = \lim_{x \to 2^+} f(x^2) = \lim_{x \to 2^+} \frac{1}{x^2 - 4}
\]

which diverges to \(+\infty\). So, indeed

\[
\lim_{x \to 2} f(g(x)) \text{ does not exist.}
\]

**Problem 8.** For \( x < 3 \), \( f(x) = cx + 1 \) which is continuous for all \( x < 3 \).

For \( x > 3 \), \( f(x) = cx^2 - 1 \) which is continuous for all \( x > 3 \).

At \( x = 3 \) both branches meet and thus we need to check the continuity from the definition:

\[
\lim_{x \to 3} f(x) = f(3).
\]

We first compute the left limit:

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} cx + 1 = 3c + 1.
\]

Then we compute the right limit:

\[
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} cx^2 - 1 = 9c - 1.
\]
So, \( \lim_{x \to 3} f(x) \) exists if and only if
\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x),
\]
which yields the equation:
\[
3c + 1 = 9c - 1.
\]
We solve and find \( c = \frac{1}{3} \). Therefore for this value of \( c \) we have
\[
\lim_{x \to 3} f(x) = 2.
\]
In order to check whether \( f(x) \) is continuous at \( x = 3 \) we compare \( \lim_{x \to 3} f(x) \) with \( f(3) \) which is computed using the first branch of the function (where \( x = 3 \) is included). So, we have
\[
f(3) = \frac{1}{3} \cdot 3 + 1 = 2,
\]
which means that indeed
\[
\lim_{x \to 3} f(x) = f(3)
\]
and so, \( f(x) \) is continuous also at \( x = 3 \). Therefore \( f(x) \) is continuous for all real values only when \( c = \frac{1}{3} \).