Midterm 1   Duration: 45 minutes
"This test has 5 questions on 8 pages, for a total of 50 points."

- Read all the questions carefully before starting to work.
- Put your final answer in the boxes provided for each question where there is an answer-box provided.
- All questions are long-answer. You should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ___________________  Last Name: ___________________

Student-No: ___________________  Section: ___________________

Signature: ___________________

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Score: ___________________

### Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

4. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:

   (i) speaking or communicating with other examination candidates, unless otherwise authorized;

   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;

   (iii) purposely viewing the written papers of other examination candidates;

   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and;

   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) (electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

5. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

6. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

7. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
1. You must show all your work in order to receive full marks for these questions. Simplify your answers as much as possible.

(a) Compute

\[ \arccos \left( \cos \left( \frac{29\pi}{3} \right) \right). \]

Answer: \( \frac{\pi}{3} \)

Solution: We let \( t := \arccos \left( \cos \left( \frac{29\pi}{3} \right) \right) \); then

\[ \cos(t) = \cos \left( \frac{29\pi}{3} \right) \]

and \( t \) is an angle in the interval \([0, \pi]\). Using the periodicity of \( \cos(x) \), we get

\[ \cos \left( \frac{29\pi}{3} \right) = \cos \left( \frac{29\pi}{3} - 10\pi \right) = \cos \left( -\frac{\pi}{3} \right). \]

However, the angle \(-\frac{\pi}{3}\) is not in the interval \([0, \pi]\) and neither is the angle \(-\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}\). On the other hand, recalling the identity

\[ \cos(-x) = \cos(x) \]

for all angles \( x \), we conclude that

\[ \cos \left( \frac{29\pi}{3} \right) = \cos \left( -\frac{\pi}{3} \right) = \cos \left( \frac{\pi}{3} \right) \]

and the angle \( \frac{\pi}{3} \) is in the interval \([0, \pi]\), which means that indeed

\[ \arccos \left( \cos \left( \frac{29\pi}{3} \right) \right) = \frac{\pi}{3}. \]

(b) Find where the following function is continuous

\[ f(x) = \sqrt{\arcsin(x)} + \pi/4. \]

Answer: \( \left[ -\frac{1}{\sqrt{2}}, 1 \right] \)

Solution: The function is continuous whenever \( \arcsin(x) + \frac{\pi}{4} \geq 0 \). We know that \( \arcsin x \) satisfies \( \arcsin x \geq -\pi/4 \) whenever \( x \geq \sin(-\pi/4) = -1/\sqrt{2} \). Finally, we note that \( \arcsin(x) \) is defined only for \( x \leq 1 \).
2. You must show all your work in order to receive full marks for these questions. Simplify your answers as much as possible.

(a) If \( f(x) = x^2 \cdot g(x) \) and \( g(3) = 5 \), while \( g'(3) = -3 \), then compute \( f'(3) \).

**Answer:** 3

**Solution:** We compute

\[
f'(x) = 2xg(x) + x^2g'(x)
\]

and so,

\[
f'(3) = 2 \cdot 3 \cdot g(3) + 3^2 \cdot g'(3) = 6 \cdot 5 - 9 \cdot 3 = 3.
\]

(b) Compute

\[
\lim_{x \to 3} \frac{x^4 - 81}{3x - x^2}
\]

**Answer:** -36

**Solution:**

\[
\begin{align*}
\lim_{x \to 3} \frac{x^4 - 81}{3x - x^2} &= \lim_{x \to 3} \frac{(x^2 - 9)(x^2 + 9)}{x(3 - x)} \\
&= \lim_{x \to 3} \frac{(x - 3)(x + 3)(x^2 + 9)}{x(3 - x)} \\
&= \lim_{x \to 3} \frac{(x - 3)(x + 3)}{x} \\
&= - \lim_{x \to 3} \frac{(x + 3)(x^2 + 9)}{x} \\
&= - \frac{(3 + 3)(9 + 9)}{3} \\
&= -36.
\end{align*}
\]
8 marks  (c) Compute the following limit:

\[ \lim_{x \to -\infty} \left( \sqrt{5x^2 + 2x} - \sqrt{5x^2 - x} \right) \]

Answer: \(-\frac{3}{2\sqrt{5}}\)

**Solution:** We multiply by the conjugate and obtain

\[ \lim_{x \to -\infty} \left( \sqrt{5x^2 + 2x} - \sqrt{5x^2 - x} \right) \cdot \frac{\sqrt{5x^2 + 2x} + \sqrt{5x^2 - x}}{\sqrt{5x^2 + 2x} + \sqrt{5x^2 - x}} \]

\[ = \lim_{x \to -\infty} \frac{(5x^2 + 2x) - (5x^2 - x)}{\sqrt{5x^2 + 2x} + \sqrt{5x^2 - x}} \]

\[ = \lim_{x \to -\infty} \frac{3x}{\sqrt{5x^2 + 2x} + \sqrt{5x^2 - x}} \]

\[ = \lim_{x \to -\infty} \frac{3}{\sqrt{\frac{5x^2 + 2x}{x^2} + \sqrt{\frac{5x^2 - x}{x^2}}} x} \]

\[ = \lim_{x \to -\infty} \frac{3}{-\sqrt{\frac{5 + \frac{2}{x}}{x} + \sqrt{\frac{5 - \frac{1}{x}}{x}}}} \]

\[ = \frac{3}{-\sqrt{5} - \sqrt{5}} \]

\[ = -\frac{3}{2\sqrt{5}}. \]
3. Find the derivative of the function \( f(x) = \sqrt{3 - 2x} \) from the definition of the derivative. If you do not compute the derivative using its definition, then you will receive 0 marks.

**Solution:** We compute

\[
\frac{f'(a)}{= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}} = \lim_{x \to a} \frac{\sqrt{3 - 2x} - \sqrt{3 - 2a}}{x - a}
\]

\[
= \lim_{x \to a} \frac{(\sqrt{3 - 2x} - \sqrt{3 - 2a}) \cdot (\sqrt{3 - 2x} + \sqrt{3 - 2a})}{(x - a) \cdot (\sqrt{3 - 2x} + \sqrt{3 - 2a})}
\]

\[
= \lim_{x \to a} \frac{(3 - 2x) - (3 - 2a)}{(x - a) \cdot (\sqrt{3 - 2x} + \sqrt{3 - 2a})}
\]

\[
= \lim_{x \to a} \frac{-2(x - a)}{(x - a) \cdot (\sqrt{3 - 2x} + \sqrt{3 - 2a})}
\]

\[
= \lim_{x \to a} \frac{-2}{\sqrt{3 - 2x} + \sqrt{3 - 2a}} = -\frac{1}{\sqrt{3 - 2a}}
\]

and so, \( f'(x) = -\frac{1}{\sqrt{3 - 2x}} \).
4. Find the slope of the tangent line at the curve given by the equation
\[ x^y = y^x \]

at the point \((4, 2)\). You must show your work.

Answer: \( \frac{\ln(2) - \frac{1}{2}}{\ln(4) - 2} = \frac{2 \ln(2) - 1}{2 \ln(4) - 4} \)

**Solution:** We apply first the logarithm and obtain
\[ y \ln(x) = x \ln(y) \]
and then differentiate with respect to \(x\):
\[ y' \cdot \ln(x) + \frac{y}{x} = \ln(x) + x \cdot \frac{y'}{y}. \]

Then substituting \(x = 4\) and \(y = 2\) yields
\[ y' \cdot \ln(4) + \frac{2}{4} = \ln(2) + 4 \cdot \frac{y'}{2} \]
and so,
\[ y' (\ln(4) - 2) = \ln(2) - \frac{1}{2} \]
and therefore, the slope of the tangent line at the point \((4, 2)\) is \( \frac{\ln(2) - \frac{1}{2}}{\ln(4) - 2} \).
5. Determine with proof the values of $a$ and $b$ for which the function

$$f(x) = \begin{cases} 
  x^3 + ax + b & \text{if } x \leq 0 \\
  x^2 \sin \left( \frac{1}{x} \right) & \text{if } x > 0 
\end{cases}$$

is differentiable at $x = 0$.

**Answer:** $a = 0$ and $b = 0$

**Solution:** From the definition, we know that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

and since, $f(0) = b$, we have that

$$f'(0) = \lim_{x \to 0} \frac{f(x) - b}{x},$$

which we compute by evaluating both the left and the right limits as we approach 0. So,

$$\lim_{x \to 0^-} \frac{f(x) - b}{x} = \lim_{x \to 0^-} \frac{x^3 + ax + b - b}{x} = \lim_{x \to 0^-} \frac{x^3 + ax}{x} = \lim_{x \to 0^-} x^2 + a = a.$$

On the other hand,

$$\lim_{x \to 0^+} \frac{f(x) - b}{x} = \lim_{x \to 0^+} \frac{x^2 \sin \left( \frac{1}{x} \right) - b}{x}.$$

Now, we first compute $\lim_{x \to 0^+} x^2 \sin(1/x)$; for this, we note that

$$-1 \leq \sin \left( \frac{1}{x} \right) \leq 1$$

and so,

$$-x^2 \leq x^2 \sin \left( \frac{1}{x} \right) \leq x^2$$

and then, taking limits as we approach 0, we note that

$$\lim_{x \to 0^+} -x^2 = 0 = \lim_{x \to 0^+} x^2$$

and therefore, using Squeeze Theorem, we obtain that also,

$$\lim_{x \to 0^+} x^2 \sin \left( \frac{1}{x} \right) = 0.$$

So, in order for the limit

$$\lim_{x \to 0^+} \frac{x^2 \sin \left( \frac{1}{x} \right) - b}{x}$$

to exist, we need that as we approach 0, the above limit becomes a limit of type $\frac{0}{0}$ (since a limit of the form $\frac{c}{0}$, where $c \neq 0$ would not exist). Therefore, we need that the numerator converges to 0, i.e.,

$$0 = \lim_{x \to 0^+} \left( x^2 \sin \left( \frac{1}{x} \right) - b \right) = -b.$$
because we already saw that \( \lim_{x \to 0^+} x^2 \sin(1/x) = 0 \); in conclusion, we must have that \( b = 0 \). So,
\[
\lim_{x \to 0^+} \frac{f(x) - b}{x} = \lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{x^2 \sin(\frac{1}{x})}{x} = \lim_{x \to 0^+} x \cdot \sin(\frac{1}{x}).
\]
The above last limit is computed again using the Squeeze Theorem since
\[
-1 \leq \sin(\frac{1}{x}) \leq 1 \quad \text{and so,} \quad -x \leq x \sin(\frac{1}{x}) \leq x
\]
and because \( \lim_{x \to 0^+} -x = 0 = \lim_{x \to 0^+} x \), we conclude that also,
\[
\lim_{x \to 0^+} x \sin(\frac{1}{x}) = 0.
\]
In conclusion, we have that
\[
\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} x \sin(\frac{1}{x}) = 0.
\]
But then, in order for \( f(x) \) to be differentiable at \( x = 0 \), we need that
\[
\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}
\]
and since we already computed that
\[
\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = 0
\]
and we’ve seen that
\[
\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^-} x^2 + a = a,
\]
we conclude that also \( a = 0 \).