PUTNAM PRACTICE SET 11

PROF. DRAGOS GHILOCA

Problem 1. Find the sum of the series
\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2n}{3^n(3^m n + 3^m m)}. \]

Solution. We let \( a_n := \frac{n}{3} \) and then we notice that our series is precisely \( S := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n^2 a_m}{a_n + a_m} \). Clearly, since the series is absolutely convergent,
\[ 2S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_n^2 a_m}{a_n + a_m} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m a_n}{a_m + a_n} = \left( \sum_{n=1}^{\infty} a_n \right)^2. \]
Now, the series \( \sum_{n=1}^{\infty} \frac{n}{3} \) represents \( f'(1) \) for the function
\[ f(x) = \sum_{n=1}^{\infty} x^n = \frac{x}{3} \cdot \frac{1}{1 - \frac{x}{3}} = \frac{x}{3 - x}. \]
So, \( f'(x) = \frac{3}{(3-x)^2} \) and therefore, \( f'(1) = \frac{4}{3} \); so, we conclude that \( S = \frac{9}{32} \).

Problem 2. Prove that there exists a positive constant \( C \) such that for any polynomial \( P \in \mathbb{R}[x] \) of degree less than 2020, we have that
\[ P(0) \leq C \cdot \int_{-1}^{1} |P(x)| \, dx. \]

Solution. First, we note that if \( P(0) = 0 \), then any positive constant \( C \) would work. So, from now on, assume \( P(0) \neq 0 \), i.e., 0 is not a root of the polynomial \( P(x) \).
Secondly, we observe that if the \( r_i \)'s are the roots of \( P(x) \) (listed with their corresponding multiplicities). So, the problem asks for proving that there exists a positive lower bound for the integral
\[ \int_{-1}^{1} \prod_{i} \left| \frac{x - r_i}{r_i} \right| \, dx. \]

Our strategy is to show that there exists a subinterval \( I \subset [-1, 1] \) of length larger than some given positive quantity such that for all points \( x \) in \( I \), each of the factors \( |(x - r_i)/r_i| \) are bounded below by another positive quantity (note that each \( r_i \) is nonzero according to our initial assumption as above).
Since \( P(x) \) has less than 2020 distinct roots, then there exists an interval \( I \subset [0, 1/2] \) of length at least \( \frac{1}{1020} \) such that none of the roots of \( P(x) \) are within \( \frac{1}{1020} \) of some point contained in \( I \).
Now, for any root $r$ of $P(x)$ and for any point $x \in I$, we claim that

$$\left| \frac{x - r}{r} \right| > \frac{1}{10^4}.$$ 

Indeed, if $|r| \leq 1$, then since $|x - r| > \frac{1}{10^4}$, then indeed $|(x - r)/r| > 1/10^4$. So, assume next that $|r| > 1$; but then

$$\left| \frac{x - r}{r} \right| = \left| 1 - \frac{x}{r} \right| \geq 1 - \left| \frac{x}{r} \right| > 1 - \frac{1}{2} = \frac{1}{10^4},$$

as claimed. So,

$$\int_{-1}^{1} \left| \frac{P(x)}{P(0)} \right| \, dx \geq \int_{I} \prod_{i} \left| \frac{x - r_i}{r_i} \right| > \int_{I} \left( \frac{1}{10^4} \right)^{2020} \, dx = \frac{1}{10^{8084}}.$$

**Problem 3.** The sequence $\{a_n\}$ satisfies

$a_1 = 1$; $a_2 = 2$; $a_3 = 24$ and for $n \geq 4$:

$$a_n = 6a_{n-1}a_{n-3} - 8a_{n-1}^2 + a_{n-2}a_{n-3}.$$

Prove that for each positive integer $n$, we have that $a_n$ is an integer multiple of $n$.

**Solution.** We let $b_n := a_n/a_{n-1}$ for each $n \geq 2$ and so, for all $n \geq 4$, we have:

$$b_n = 6b_{n-1} - 8b_{n-2},$$

where

$b_2 = 2$ and $b_3 = 12$.

We solve first for the sequence $\{b_n\}$ whose characteristic roots are 2 and 4 and a simple computation yields that for all $n \geq 2$, we have:

$$b_n = -2^{n-1} + 4^{n-1}.$$

So, using that $a_1 = 1$, we conclude that

$$a_n = \prod_{i=1}^{n-1} (4^i - 2^i).$$

Now, for each positive integer $n$, we write it as $n = 2^a \cdot b$, where $a \geq 0$ and $b$ is an odd integer. We have that, after denoting by $\phi(m)$ the Euler-totient function corresponding to each integer $m$,

$$4^{a \phi(b)} - 2^{a \phi(b)} \equiv 0 \pmod{n}.$$

Indeed, clearly, the above expression is divisible by $2^a$, so we’re left to prove that it must also be divisible by $b$. However,

$$4^{a \phi(b)} - 2^{a \phi(b)} = 2^{a \phi(b)} \left( 2^{a \phi(b)} - 1 \right) \equiv 0 \pmod{b},$$

using Euler’s theorem because $2^{\phi(b)} \equiv 1 \pmod{b}$ (and then $2^{m \phi(b)} \equiv 1 \pmod{b}$ for any positive integer $m$). Finally, we observe that

$$a \cdot \phi(b) < n = 2^a \cdot b,$$

because $\phi(b) \leq b$ and $a < 2^a$ for any $b \geq 1$ and any $a \geq 0$.

**Problem 4.** Let $P \in \mathbb{C}[x]$ be a polynomial of degree $n$ such that $P(x) = Q(x) \cdot P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''$ is the double derivative of
$P$. Show that if $P(x)$ has at least two distinct roots, then it must have $n$ distinct roots.

**Solution.** Assume $r$ is a root of $P(x)$ of multiplicity $m \geq 2$. Then $P''(x)$ has the root $r$ with multiplicity $m - 2$; therefore, $Q(x)$ must have the root $r$ with multiplicity 2. Furthermore, looking the leading coefficients of both $P(x)$ and of $P''(x)$, we conclude that $Q(x) = \frac{f}{m(n-1)} \cdot (x - r)^2$. Now, we write

$$P(x) = \sum_{i=0}^{n} c_i (x - r)^i;$$

actually, from our assumption, we know that $a_i = 0$ for $0 \leq i < m$ (where $m \geq 2$). Then

$$P''(x) = \sum_{i=m}^{n} i(i - 1)c_i(x - r)^{i-2}$$

and then equating $P(x) = \frac{(x-r)^2}{m(n-1)} \cdot P''(x)$ (in their expansions around $x = r$), we get that $c_i$ must be equal to 0 whenever $i < n$, which contradicts the assumption that $P(x)$ has at least two distinct roots. This concludes our proof.