Problem 1. Find the largest possible integer which is the product of finitely many positive integers whose sum equals 2018.

Solution. Let $x_1, \ldots, x_r$ be positive integers whose sum is 2018 and which have the largest possible product.

First we notice that if $x_r \geq 4$, then replacing $x_r$ by $x'_r = 2$ and $x'_{r+1} = x_r - 2$, while $x'_i = x_i$ for each $i \leq r - 1$ leads to a larger product. So, this means that each $x_i$ is less than 4.

Secondly, we cannot have $x_r = 1$ since then replacing $x'_{r-1} = x_r - 1$ and keeping $x'_i = x_i$ for each $i \leq r - 2$ would lead to a sequence $x'_1, \ldots, x'_{r-1}$ whose sum is 2018 but whose product is larger than for the product of the original numbers $x_i$.

So, we conclude that each $x_i \in \{2, 3\}$. Now, if we were to have three of the $x_i$'s equal to 2, we could replace them with two numbers equal to 3 and the product would only increase. Therefore, we have only one or two numbers equal to 2 and all the rest of the numbers equal 3. Since 2018 $\equiv 2 \pmod{3}$, this means $x_1 = 2$ and $x_i = 3$ for each $i = 2, \ldots, r$; clearly, since $2 + 3r = 2018$, then we must have $r = 672$. So, the largest product of the numbers adding up to 2018 is $2 \cdot 3^{672}$.

Problem 2. Let $P \in \mathbb{R}[x]$ be a polynomial with the property that $P(x) > 0$ for each positive real number $x$. Then prove that there exist polynomials $Q_1, Q_2 \in \mathbb{R}[x]$ with all coefficients nonnegative, such that $P = Q_1 \cdot Q_2$.

Solution. First of all, since $P(x) > 0$ for all $x > 0$, we conclude that its leading coefficient must be positive; so, without loss of generality we may assume from now on that $P(x)$ is monic since its leading coefficient can be absorbed in $Q_1(x)$.

Second, we know that $P(x)$ is a product of linear polynomials of the form $x + r_i$ for some nonnegative real numbers $r_i$ and perhaps also a product of unfactorable quadratics (over $\mathbb{R}$), i.e., quadratics of the form $x^2 + a_i x + b_i$ where $a_i^2 < 4b_i$. So, it suffices to prove that each polynomial of the form

$$x + r_i \text{ for some } r_i \geq 0, \text{ and}$$

$$x^2 + a_i x + b_i \text{ where } a_i^2 < 4b_i.$$

is of the form $\frac{Q_1(x)}{Q_2(x)}$ where each $Q_1, Q_2$, are polynomials with nonnegative real coefficients. Clearly, this statement holds for polynomials of the form $x + r_i$; so, we’re left to analyze the case of quadratic polynomials. In this latter case, we let $f(x) := x^2 + ax + b$

such a quadratic polynomial with $a^2 < 4b$; then we let $b := r^2$ for some positive real number $r$ and then we let $t \in [0, \pi]$ such that $a = -2r \cos(t)$. Our goal is to
find some polynomials \( g_1(x) \) and \( g_2(x) \) with nonnegative real coefficients such that 
\[
f(x) = \frac{g_1(x)}{g_2(x)},
\]
If \( t \in [\pi/2, \pi] \), then we are done (simply take \( g_1(x) := f(x) \) and \( g_2(x) := 1 \)).

Now, if \( t \in [0, \pi/2) \) (i.e., \( \cos(t) > 0 \) and implicitly, \( a < 0 \)), we observe that
\[
f(x) \cdot (x^2 - ax + b) = x^4 - (a^2 - 2b)x^2 + b^2 = x^4 - 2a^2 \cos(2t) + r^4.
\]

Then we repeat our analysis and so, if \( 2t \in [\pi/2, \pi] \), then we are done since then \( \cos(2t) \leq 0 \). Now, if \( 2t \in [0, \pi/2) \) (and so, implicitly, \( a^2 > 2b \)), then we repeat the construction and get:
\[
f(x) \cdot (x^2 - ax + b) \cdot (x^4 + (a^2 - 2b)x^2 + b^2) = x^8 - 2a^2 \cos(4t)x^4 + r^8.
\]
Eventually, there must exist a first positive nonnegative integer \( i_0 \) such that \( 2^{i_0}t \in [\pi/2, \pi] \) and for that \( i_0 \), we have that the corresponding polynomial
\[
x^{2^i_0 + 1} - 2a^2 \cos(2^{i_0}t)x^{2^i_0} + r^{2^{i_0} + 1}
\]
has all its coefficients nonnegative and we reached this polynomial by multiplying \( f(x) \) by polynomials which were themselves with nonnegative coefficients.

**Problem 3.** Prove that there exist infinitely many \( n \in \mathbb{N} \) with the property that 
\( 7^n \) contains in its decimal expansion 2018 consecutive digits equal to 0.

**Solution.** The point is that \( \gcd(7, 10) = 1 \) and so, Euler’s Theorem guarantees that
\[
7^{2018} \cdot 2^{2020} \equiv 7^{\phi(10^{2019})} \equiv 1 \pmod{10^{2019}},
\]
thus showing that \( 7^{2018} \cdot 2^{2020} \) ends with the digit 1 and it has 2018 digits of 0 preceding that last digit.

**Problem 4.** Let \( a \in (0, 1) \) be a real number. We consider the function \( f : (0, 1] \to (0, 1] \) given by:
\[
f(x) = x + 1 - a \quad \text{if } 0 < x \leq a \quad \text{and} \quad f(x) = x - a \quad \text{if } a < x \leq 1.
\]
Prove that for any interval \( I \subseteq (0, 1] \), there exists a positive integer \( n \) such that
\[f^{\circ n}(I) \cap I \neq \emptyset.
\]

**Solution 1.** We note that \( f \) is a bijection mapping \( (0, 1] \) into itself. Also, we claim that for any interval \( J \subseteq (0, 1] \), we have that \( f(J) \) is also a union of intervals whose sums of their lengths equals the length of \( J \). This is proven easily by considering the three cases:

**Case 1.** \( J \subseteq (0, a] \). In this case, \( f(J) \) is an interval of the same length as \( J \) contained in \((1 - a, 1)\].

**Case 2.** \( J \subseteq (a, 1] \). In this case, \( f(J) \) is an interval of the same length as \( J \) contained in \((0, 1 - a)\].

**Case 3.** \( J = (\alpha, \beta] \) (or any other choice of including or not any of the two endpoints) for some \( 0 \leq \alpha < a < \beta \leq 1 \). In this case, \( f(J) = (\alpha + 1 - a, 1] \cup (0, \beta - a] \) whose length is 
\[
1 - (\alpha + 1 - a) + (\beta - a) - 0 = \beta - \alpha, \text{ as claimed.}
\]

Now, if \( f^m(I) \cap I = \emptyset \), then we claim that \( f(I) \cap f^j(I) = \emptyset \) for any integers \( i > j \geq 0 \). Indeed, using the fact that \( f \) is a bijection on \((0, 1] \) (and therefore, \( f^m \) is a bijection for each \( m \in \mathbb{N} \)), we get that if there exists some \( x \in f(I) \cap f^j(I) \), then
letting $y \in (0, 1]$ be the unique real number such that $f^i(y) = x$, then we would have that $y \in I \cap f^{-j}(I)$, contradiction. (Note that we do not claim that $y$ is fixed by $f^{-j}$, however we know that $x = f^j(y) \in f^j(f^{-j}(I))$ and $f^j$ is a bijection, thus showing that $y \in f^{-j}(I)$.) But then we would have an infinite sequence of unions of intervals $f^n(I)$, each one of them of total length equal to the length of $I$ and all these intervals would fit into the interval $(0, 1]$, which is a contradiction. So, indeed there must be some $n \in \mathbb{N}$ such that $f^n(I) \cap I \neq \emptyset$.

Solution 2. We notice that from our definition of the function $f$, we have that for any real number $x$, we have that $f(x) = x + n a \in \mathbb{Z}$. By induction, we prove that $f^n(x) = x + n a \in \mathbb{Z}$ for each $x \in (0, 1]$. Indeed, assuming that there exists some $p_n(x) \in \mathbb{Z}$ (i.e., an integer depending on $x$) such that

$$f^n(x) = x - na + p_n$$

then we compute

$$f^{n+1}(x) = f(x - na + p_n(x)) = x - na + p_n(x) - a + p_1(x - na + p_n(x)) \in (x - (n+1)a + \mathbb{Z},$$

where $p_1(x) := f(x) - (x - a)$ (and more generally, $p_{n}(x) := f^{n}(x) - (x - na)$). So, indeed, $f^n(x) = x - na \in \mathbb{Z}$ for each $n \in \mathbb{N}$ and for each $x \in (0, 1]$.

Now, for any given interval $I$ we claim that there must exist some $x \in I$ such that also $x \in f^n(I)$, i.e., there exists some $y \in I$ such that

$$x = f^n(y) = y - na + p_n(y).$$

So, $na - p_n(y) = y - x$, i.e., for any $\epsilon > 0$, there exists some positive integer $n$ and some integer $q_n$ such that $na - q_n \in (-\epsilon, \epsilon)$. The conclusion follows from a classical argument looking at the fractional part of $na$ as we vary $n \in \mathbb{N}$ and note that for some $N$ sufficiently large (anything larger than $1/\epsilon$ would work) we must have two distinct integers $N \geq i > j \geq 0$ such that $|\{ia\} - \{ja\}| < \epsilon$ and so, $(i - j)a - q \in (-\epsilon, \epsilon)$, where $q = \lfloor ia \rfloor - \lfloor ja \rfloor$ (the difference of their corresponding integer parts).

Problem 5. Find (with proof) all possible function $f : \mathbb{N} \rightarrow \mathbb{N}$ with the property that $f(n + 1) > f(f(n))$ for each $n \in \mathbb{N}$.

Solution. We will prove that there is only one such function, which is $f(n) = n$ for each $n \in \mathbb{N}$.

First we prove by induction on $k$ that for each $n \geq k$, we have that $f(n) \geq k$. The base case $k = 1$ is obvious. So, assuming that we prove $f(n) \geq k$ for each $n \geq k$, next we derive that $f(n) \geq k + 1$ for each $n \geq k + 1$. Indeed, for any $n \geq k$, we have that

$$f(n + 1)$$

$$> f(f(n))$$

by the main hypothesis

$$= f(m)$$

for some $m \geq k$ since $n \geq k$ and using the inductive hypothesis

$$\geq k$$

again by the inductive hypothesis.

So, indeed $f(n + 1) \geq k + 1$ for each $n \geq k$, which concludes the proof for our claim that $f(n) \geq k$ whenever $n \geq k$ for any given $k \in \mathbb{N}$.

Now, assume there exists some $n \in \mathbb{N}$ such that $f(n) > n$ and we will derive a contradiction, which will conclude our proof that the only function is the one
satisfying $f(n) = n$ for each $n \in \mathbb{N}$. So, let $n_1$ be the smallest positive integer $n$ such that $f(n) > n$. Clearly, we cannot have $f(n_1) = n_1 + 1$ since then
\[ f(n_1 + 1) > f(f(n_1)) = f(n_1 + 1), \] contradiction.
Also, since $n_1$ is the smallest such positive integer, then it must be that for each positive integer $n < n_1$, we have that $f(n) = n$. Now, for each $n > n_1$, we have that $f(n) > f(f(n - 1))$ and moreover, $f(n - 1) \geq n_1$ since $n > n_1$ and $f(k) \geq k$ for each $k \in \mathbb{N}$. Now, if $f(n - 1) = n_1$, we note that it cannot be that $n - 1 \geq n_1 + 1$ since then we would have that $f(n - 1) \geq n_1 + 1$, a contradiction. So, if $f(n - 1) = n_1$ then we would get that $n - 1 = n_1$, which is again a contradiction since our assumption yields that $f(n_1) > n_1$. In conclusion, we must have that $f(n - 1) > n_1$. So, this means that our hypothesis that $f(n_1) > n_1$ yields the following property: for each $n > n_1$, there exists some $m > n_1$ such that $f(n) > f(m)$ (more precisely, $m = f(n - 1)$). But this would mean that the set of positive integers
\[ \{f(n_1 + 1), f(n_1 + 2), \ldots, \ldots \} \]
does not have a minimal element, which is impossible. So, indeed, we must have that $f(n) = n$ for each $n \in \mathbb{N}$. 