Math 340 Practice Questions - Solutions

Problem 1. Set up a cutting-stock problem for a paper mill that produces 100-inch raw rolls of paper, and wants to minimize the number of these needed to produce the following final quantities:

- 10 rolls that are 57 inches
- 20 rolls that are 40 inches
- 25 rolls that are 22 inches

You should set this up with inequality constraints and only using “non-wasteful” cutting patterns; you should need five variables. (You do not need to actually solve this problem!)

Solution. We just need to find what the possible cutting patterns are, which is more an exercise in logic than in linear programming. We could think through it as follows:

- Two 57-inch rolls can’t fit on a single 100-inch one, so any cutting pattern will have either 0 or 1.
- Suppose our pattern begins with one 57-inch roll. Then we have 43 inches of leftover, which can accommodate at most one 40-inch roll.
  - We could have one 57-inch roll and one 40-inch roll, with 3 inches left so we can’t fit anything else. This pattern will be \( x_1 \).
  - We could have one 57-inch roll and zero 40-inch rolls, and then cram in as many 22-inch rolls as would fit (which is just one). This pattern is \( x_2 \).
- We can also look at patterns with no 57-inch rolls. These can contain either zero, one, or two 40-inch rolls, and as much of the leftover as possible then needs to be used up with 22-inch rolls.
  - A pattern with two 40-inch rolls can’t contain any 22-inch ones. This is \( x_3 \).
  - A pattern with one 40-inch rolls can have two 22-inch rolls cut out of the rest; this will be \( x_4 \).
  - A pattern with no 40-inch rolls can have four 22-inch rolls; this will be \( x_5 \).

With our patterns set, we just need to set up the corresponding LP (or really, integer programming problem since you can’t cut a fractional roll!)

- Minimize \( x_1 + x_2 + x_3 + x_4 + x_5 \), subject to:
  - \( x_1 + x_2 \geq 10 \) (the 57-inch rolls we need)
  - \( x_1 + 2x_3 + x_4 \geq 20 \) (the 40-inch rolls we need)
  - \( x_2 + 2x_4 + 4x_5 \geq 25 \) (the 22-inch rolls we need).
  - \( x_1, x_2, x_3, x_4, x_5 \geq 0 \) are integers.

Problem 2. Consider the following integer programming problem.

- Maximize \( x_2 \), subject to:
  - \( -x_1 + x_2 \leq 1 \)
  - \( 3x_1 + x_2 \leq 7 \)
  - \( x_1, x_2 \geq 0 \) integers

(If you want to get a geometric idea of what’s going on as you solve this, it may help to graph it!)
(a) Solve the relaxed LP for this problem (i.e. without the integer constraints). The optimum should have both \( x_1, x_2 \) as non-integers.

**Solution.** This is just running the simplex method. We start with

\[
\begin{align*}
x_3 &= 1 + x_1 - x_2 \\
x_4 &= 7 - 3 x_1 - x_2 \\
z &= \frac{1}{2} x_2
\end{align*}
\]

Then \( x_2 \) enters and \( x_3 \) leaves:

\[
\begin{align*}
x_2 &= 1 + x_1 - x_3 \\
x_4 &= 6 - 4 x_1 + x_3 \\
z &= \frac{1}{2} + x_1 - x_3
\end{align*}
\]

Next \( x_1 \) enters and \( x_4 \) leaves:

\[
\begin{align*}
x_2 &= \frac{5}{2} - \frac{1}{2} x_4 - \frac{3}{4} x_3 \\
x_1 &= \frac{1}{4} - \frac{3}{4} x_4 + \frac{1}{4} x_3 \\
z &= \frac{5}{2} - \frac{1}{4} x_4 - \frac{3}{4} x_3
\end{align*}
\]

This is optimal, i.e. the optimal solution is \( z = 5/2 \) at \( x_1 = 3/2, x_2 = 5/2 \).

(b) Use the “branching procedure” on one of the two variables (either one will work), setting up two LPs such that the optimum integer solution to our original problem must arise from one of them, but they exclude the optimum general solution found in part (a). Use these LPs to find the integer optimum.

**Solution.** Say we want to branch on \( x_1 \). This requires looking at two distinct LPs:

- Maximize \( x_2 \) subject to:
  - \(-x_1 + x_2 \leq 1\)
  - \(3 x_1 + x_2 \leq 7\)
  - \(x_1 \leq 1\)
  - \(x_1, x_2 \geq 0\) integers

  and

- Maximize \( x_2 \) subject to:
  - \(-x_1 + x_2 \leq 1\)
  - \(3 x_1 + x_2 \leq 7\)
  - \(x_1 \geq 2\)
  - \(x_1, x_2 \geq 0\) integers

Let’s start with the first one. This adds a new slack variable \( x_5 = 1 - x_1 \) to the initial dictionary, and thus the final dictionary from (a) becomes

\[
\begin{align*}
x_2 &= \frac{5}{2} - \frac{1}{2} x_4 - \frac{3}{4} x_3 \\
x_1 &= \frac{1}{4} - \frac{3}{4} x_4 + \frac{1}{4} x_3 \\
x_5 &= \frac{1}{2} + \frac{3}{4} x_4 - \frac{1}{4} x_3 \\
z &= \frac{5}{2} - \frac{3}{4} x_4 - \frac{3}{4} x_3
\end{align*}
\]
We thus need to do a pivot of the dual simplex method, letting $x_5$ leave and $x_4$ enter:

\[
\begin{align*}
    x_2 &= 2 -x_5 -x_3 \\
    x_1 &= 1 -x_5 \\
    x_4 &= 2 +4x_5 +x_3 \\
    z &= 2 -x_5 -x_3
\end{align*}
\]

So the optimum value is $z = 2$ at $x_1 = 1$, $x_2 = 2$, which is an integer solution. And actually, this has to be an optimum integer solution to the entire problem; since our optimal general solution was $z = 5/2$, the best integer solution we could hope for is the next smallest integer $2$. So we don’t even need to solve the other LP since we know that its optimum integer solution can’t beat this one!

**Problem 3.** (From a previous exam) Players $A$ and $B$ play a game where they simultaneously call out either the number “one” or “two”. Player $A$ (row player) wins if the sum of the numbers is odd. Player $B$ (column player) wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the two numbers.

(a) What is the payoff matrix for this game?

**Solution.** Letting the first row and column be “one” and the second row and column “two” the matrix is

\[
\begin{bmatrix}
    -2 & 3 \\
    3 & -4
\end{bmatrix};
\]

in each case the magnitude of the entry is the sum of the numbers called out, and it is positive (goes to player $A$) when it’s odd and negative (to player $B$) when it’s even.

(b) Find the optimal strategies of both players.

**Solution.** Let’s start with Player $A$. Remember the LP we want to set up is maximizing $z = \min \mathbf{x}^T M$ over $\mathbf{x}$ a stochastic vector, which in this case can be written as:

- Maximize $z$, subject to:
  - $z \leq -2x_1 + 3x_2$
  - $z \leq 3x_1 - 4x_2$
  - $x_1 + x_2 = 1$
  - $x_1, x_2 \geq 0$, $z$ free

To use the simplex method on this, we need to put it into standard form. One way to do that is to split the equality into two constraints, split the free variable $z$ into two variables, and proceed. We can use a few tricks to simplify this, though:

- Rearrange $x_1 + x_2 = 1$ to $x_2 = 1 - x_1$ and substitute that in everywhere to eliminate $x_2$. (Note the constraint $x_2 \geq 0$ turns into $x_1 \leq 1$).
- Assume $z$ actually satisfies $z \geq 0$. (Since the objective function is to maximize the variable $z$, either the optimum will occur with $z \geq 0$ and we’ll find it, or the optimum will occur when $z < 0$ and we’ll get an infeasible situation by assuming $z \geq 0$). This will turn out to work in this case, though just as a warning you need to be really careful when doing this sort of thing (but it makes sense if the objective function equals a single variable like $z$ and you suspect the optimum might be positive).

So we’ll look at the system
• Maximize $z$, subject to:
  o $z + 5x_1 \leq 3$
  o $z - 7x_1 \leq -4$
  o $x_1 \leq 1$
  o $x_1 \geq 0$, and making an assumption $z \geq 0$

Solving this via the simplex method gives an optimum of $z = 1/12$ when $x_1 = 7/12$ and thus $x_2 = 5/12$. This is the optimal solution for player $A$.

For player $B$, we could set up and solve another LP, or we could use the minimax theorem to tell us that player $B$’s optimum must be some $[y_1 \ y_2]^\top$ such that the maximum entry of $M\vec{y}$ is exactly the value $1/12$ we found for player $A$. We have

$$M\vec{y} = \begin{bmatrix}
-2 & 3 \\
3 & -4 \\
\end{bmatrix} \begin{bmatrix}
y_1 \\
y_2 \\
\end{bmatrix} = \begin{bmatrix}
-2y_1 + 3y_2 \\
3y_1 - 4y_2 \\
\end{bmatrix}.$$ 

We find that if we plug in $y_1 = 7/12$ and $y_2 = 5/12$ both entries of this become equal to $1/12$, so this is optimal. (I got this solution by guessing that I’d be able to find something making both entries equal to $1/12$ and solving the resulting system of two equations!)

(c) What is the value of this game? Which player has the advantage?

Solution. The value is the $1/12$ we got from finding the optimum strategies in part (b). This means that player $A$ has the advantage, since they expect to win an average of $1/12$ per game (and accordingly $B$ expects to lose an average of $1/12$ per game).

Problem 4. (From a previous exam) A matrix $M$ is called doubly stochastic if all of its entries are non-negative, every row adds up to 1, and every column adds up to 1; an example is

$$\begin{bmatrix}
7/12 & 0 & 5/12 \\
1/6 & 1/2 & 1/3 \\
1/4 & 1/2 & 1/4 \\
\end{bmatrix}.$$ 

Prove that if $M$ is an $n \times n$ doubly stochastic matrix, then $v(M) = 1/n$, that is, the value of the game with payoff matrix $M$ is equal to $1/n$.

Solution. We claim that if we take $\vec{x}$ and $\vec{y}$ to be the vectors with $1/n$ in each entry (i.e. the strategy of picking each option with the same probability), then this is optimal. Recall that all we need to show here is that the minimum value of $\vec{x}^\top M$ equals the maximum value of $M\vec{y}$ equals $1/n$, since then a theorem from class tells us that these are both optimal and the value of the came is $1/n$.

But this is basically immediate from the definition: each entry of $\vec{x}^\top M$ is $1/n$ times the corresponding column sum (which equals 1), and each entry of $M\vec{y}$ is $1/n$ times the corresponding row sum (which equals 1).