Midterm (Version 1) Solutions

Problem 1. [30 points] Solve the following linear program by using the two-phase simplex method and the standard rule. You should have to do a pivot to feasibility and one more pivot in Phase 1, and one pivot in Phase 2.

- Maximize \(-2x_1 - 2x_2\), subject to:
  - \(2x_1 + 2x_2 + x_3 \leq 8\)
  - \(-x_1 - x_2 - x_3 \leq -3\)
  - \(-x_1 \leq -2\)
  - \(x_1, x_2, x_3 \geq 0\)

We start by setting up the dictionary for this LP:

\[
\begin{align*}
x_4 &= 8 -2x_1 -2x_2 -x_3 \\
x_5 &= -3 +x_1 +x_2 +x_3 \\
x_6 &= -2 +x_1 \\
z &= -2x_1 -2x_2
\end{align*}
\]

Since this is an infeasible initial dictionary we have to go to the auxiliary problem.

\[
\begin{align*}
x_4 &= 8 -2x_1 -2x_2 -x_3 +x_0 \\
x_5 &= -3 +x_1 +x_2 +x_3 +x_0 \\
x_6 &= -2 +x_1 +x_0 \\
z' &= -x_0
\end{align*}
\]

Our pivot to feasibility is having \(x_0\) enter and \(x_5\) (the most infeasible variable) leave:

\[
\begin{align*}
x_4 &= 11 -3x_1 -3x_2 -2x_3 +x_5 \\
x_0 &= 3 -x_1 -x_2 -x_3 +x_5 \\
x_6 &= 1 -x_2 -x_3 +x_5 \\
z' &= -3 +x_1 +x_2 +x_3 -x_5
\end{align*}
\]

Now, the standard rule tells us to have \(x_1\) enter and \(x_0\) leave, so we get

\[
\begin{align*}
x_4 &= 2 +3x_0 +x_3 -2x_5 \\
x_1 &= 3 -x_0 -x_2 -x_3 +x_5 \\
x_6 &= 1 -x_2 -x_3 +x_5 \\
z' &= -x_0
\end{align*}
\]

We’ve reached a point where \(x_0\) is nonbasic in a feasible dictionary, so this means we’re done with phase 1, and we can just drop the \(x_0\)’s to get a feasible dictionary for the original problem. We have to fill in the objective function in terms of the nonbasic variables:

\[
\begin{align*}
x_4 &= 2 +x_3 -2x_5 \\
x_1 &= 3 -x_2 -x_3 +x_5 \\
x_6 &= 1 -x_2 -x_3 +x_5 \\
z &= -6 +2x_3 -2x_5
\end{align*}
\]

(continued on next page)
Now we proceed with Phase 2, the simplex method as usual. We pick $x_3$ to enter and then $x_6$ to leave, giving us

\[
\begin{align*}
    x_4 &= 3 -x_2 -x_6 -x_5 \\
    x_1 &= 2 +x_6 \\
    x_3 &= 1 -x_2 -x_6 +x_5 \\
    z &= -4 -2x_2 -2x_6
\end{align*}
\]

This is an optimal dictionary, so our original LP has an optimal value of $z = -4$ at $x_1 = 2$, $x_2 = 0$, $x_3 = 1$. 
Problem 2. [30 points] (a) Write down the dual to the following linear program.

- Maximize $2x_1 - 5x_2 + x_3$ subject to
  - $x_1 + x_2 + x_3 \leq 5$
  - $x_1 - 2x_2 \leq 3$
  - $x_2 + 3x_3 \leq 8$
  - $x_1, x_2, x_3 \geq 0$

The dual is

- Minimize $5y_1 + 3y_2 + 8y_3$ subject to
  - $y_1 + y_2 \geq 2$
  - $y_1 - 2y_2 + y_3 \geq -5$
  - $y_1 + 3y_3 \geq 1$
  - $y_1, y_2, y_3 \geq 0$

(b) The vector $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = (3, 0, 2)$ is optimal for the primal linear program in part (a). Use complementary slackness applied to this solution to find a vector $\mathbf{y}^*$ that is optimal for the dual. (You should not use the simplex method on the dual problem!)

Suppose $\mathbf{y}^* = (y_1^*, y_2^*, y_3^*)$ is optimal for the dual. Since the first and third variables of our optimal solution for the primal are nonzero, the first and third constraints of the primal must have no slack, i.e. be equalities. Also plugging in the values of $\mathbf{x}^*$ into the constraints for the primal we find the first and second are equalities but the third constraint is $x_2^* + 3x_3^* = 6 < 8$; since this slack the third variable of the dual must be zero. Collecting this we've said our dual solution satisfies

$$y_1^* + y_2^* = 2 \quad y_1^* + 3y_3^* = 1 \quad y_3^* = 0.$$ 

It's then easy to solve that $y_1^* = y_2^* = 1$, so our optimal dual solution must have been $\mathbf{y}^* = (1, 1, 0)$. (And sure enough, we can check that this is feasible and thus the complementary slackness theorem certifies for us that $\mathbf{x}^*$ and $\mathbf{y}^*$ are both optimal for their problems).

(c) What does the strong duality theorem tell should be true about the optimal solutions $\mathbf{x}^*$ and $\mathbf{y}^*$ from part (b)? Compute the values to check that this is indeed true.

Strong duality says that the optimum value of the primal (i.e. the value of its objective function at $\mathbf{x}^*$) equals the optimum value of the dual (the value of its objective function at $\mathbf{y}^*$). And sure enough we can compute both values are equal to 8:

$$2x_1^* - 5x_2^* + x_3^* = 2 \cdot 3 - 5 \cdot 0 + 2 = 8,$$

$$5y_1^* + 3y_2^* + 8y_3^* = 5 \cdot 1 + 3 \cdot 1 + 8 \cdot 0 = 8.$$
Problem 3. [20 points] (a) Suppose we have two linear programs with the same constraints but different objective functions. Is it possible for one of them to be infeasible and the other to have an optimal solution? Either give an example or explain why not.

This is impossible, because infeasibility of an LP only has to do with whether the constraints can be satisfied; the objective function is irrelevant when talking about feasibility.

(b) Again suppose we have two linear programs with the same constraints but different objective functions. Is it possible for one of them to be unbounded and the other to have an optimal solution? Either give an example or explain why not.

This is possible, if the feasible region is an unbounded set. A simple example is the region defined by the constraints $x_1 \geq 0, x_2 \geq 0$, which is just the upper-right quadrant of the plane. On this region the LP

- Maximize $x_1 + x_2$ subject to $x_1, x_2 \geq 0$

is unbounded (we can make this objective function as large as we want by increasing $x_1$ and/or $x_2$ since there’s no constraint on doing that) but

- Maximize $-x_1 - x_2$ subject to $x_1, x_2 \geq 0$

has an optimal solution of $x_1 = x_2 = 0$ (since making $x_1$ and/or $x_2$ large just decreases the objective function, and we can’t make them any smaller than 0).
Problem 4. [20 points] Let 0 denote a vector with all entries 0 and −1 denote a vector with all entries −1. For any matrix A, prove that exactly one of the following statements is true.

1. There exists \( \vec{x} \geq 0 \) with \( A\vec{x} \leq -1 \).
2. There exists \( \vec{y} \geq 0 \) with \( A^\top \vec{y} \geq 0 \) and \( \vec{y} \neq 0 \).

We set up a primal and dual pair of LPs as follows:

- **Primal:** Maximize \( 0 \cdot \vec{x} \) subject to \( A\vec{x} \leq -1 \) and \( \vec{x} \geq 0 \).
- **Dual:** Minimize \( -1 \cdot \vec{y} \) subject to \( A^\top \vec{y} \geq 0 \) and \( \vec{y} \geq 0 \).

Note that the dual is feasible because \( \vec{y} = 0 \) is a solution, so there are two possibilities: either both the primal and dual have optimal solutions, or the primal is infeasible and the dual is unbounded.

*The primal is infeasible and the dual is unbounded.* In this case (1) is false because the primal is infeasible, and (2) is true because we can just pick any feasible solution \( \vec{y} \) to the dual with objective value \( \leq -1 \), say.

*Both have an optimal solution.* In this case condition (1) holds because the primal is feasible. On the other hand, the optimum value of the primal is obviously 0, so the optimum value of the dual is also zero. But since the objective function for the dual is \( -1 \cdot \vec{y} = -y_1 - y_2 - \cdots - y_m \), any vector \( \vec{y} \neq 0 \) would have strictly negative objective value; so the optimum being 0 means \( \vec{y} = 0 \) is the only feasible solution for the dual. Thus (2) is false.