Math 340 Lecture 5

Summary of the simplex method. Let’s go back and look at what we did in our first example of the simplex method last time. We started with an LP in standard form:

- Maximize \( 6x_1 + 8x_2 + 5x_3 + 9x_4 \) subject to:
  - \( 2x_1 + x_2 + x_3 + 3x_4 \leq 5 \)
  - \( x_1 + 3x_2 + x_3 + 2x_4 \leq 3 \)
  - \( x_1, x_2, x_3, x_4 \geq 0 \).

We introduced slack variables and got a first dictionary

\[
\begin{align*}
x_5 &= 5 -2x_1 - x_2 - x_3 - 3x_4 \\
x_6 &= 3 -x_1 -3x_2 -x_3 -2x_4 \\
z &= 6x_1 + 8x_2 + 5x_3 + 9x_4 \\
\end{align*}
\]

this dictionary is associated to the feasible solution \( x_1 = x_2 = x_3 = x_4 = 0, x_5 = 5, x_6 = 3 \), with value \( z = 0 \). In this dictionary the slack variables \( x_5 \) and \( x_6 \) are on the left-hand side of equations (we call them basic variables) and the original variables \( x_1, x_2, x_3, x_4 \) are on the right-hand sides (we call them non-basic variables). Each iteration of the simplex method performs a pivot: for the first step we can choose \( x_1 \) as the “entering variable” and \( x_5 \) as the “leaving variable” for the dictionary. (Remark: If the words “basic” and “pivot” make you think of linear algebra, they should, and we’ll get back to that connection later).

Doing the algebra gives us a second dictionary

\[
\begin{align*}
x_1 &= \frac{5}{7} - \frac{1}{7}x_2 - \frac{1}{7}x_3 - \frac{3}{7}x_4 - \frac{1}{7}x_5 \\
x_6 &= \frac{1}{7} - \frac{5}{7}x_2 - \frac{1}{7}x_3 - \frac{3}{7}x_4 + \frac{5}{7}x_5 \\
z &= 15 + 5x_2 + 2x_3 - 3x_5 \\
\end{align*}
\]

with \( x_1, x_6 \) as the basic variables, and associated to the feasible solution \( x_1 = 5/2, x_2 = x_3 = x_4 = x_5 = 0 \), and \( x_6 = 1/2 \), with value \( z = 15 \). We do another iteration, pivoting by choosing \( x_3 \) as the entering variable and \( x_6 \) as the leaving variable. This gives us a third dictionary

\[
\begin{align*}
x_1 &= 2 + 2x_2 - x_4 - x_5 + x_6 \\
x_3 &= 1 - 5x_2 - x_4 + x_5 - 2x_6 \\
z &= 17 - 5x_2 - 2x_4 - x_5 - 4x_6 \\
\end{align*}
\]

Here \( x_1, x_3 \) are the basic variables, and the feasible solution we can read off is \( x_1 = 2, x_2 = 0, x_3 = 1, x_4 = x_5 = x_6 = 0 \), with value \( z = 17 \). Moreover, since all of the coefficients in our formula for \( z \) are negative, this tells us that 17 is the absolute maximum \( z \) can attain (because all of the \( x_i \)'s have positivity constraints!) and we’ve found our optimal solution.

So a brief summary of the simplex method so far, for a linear program on \( x_1, \ldots, x_n \) with \( m \) constraints:

- Insert slack variables \( x_{n+1}, \ldots, x_{n+m} \) so our constraints are positivity constraints for every variable, plus a list of \( m \) equations each writing one of the slack variables in terms of the original variables. This gives us a starting dictionary with the slack variables as the basic ones, and the original variables as the non-basic ones.
- If the original dictionary is feasible (i.e. taking the non-basic variables equal to zero gives a feasible solution) proceed. If not, ??????
- Start pivoting; with each iteration pick an entering variable that has a positive coefficient in \( z \), and pick an exiting variable that preserves feasibility (keeps all of the constant terms \( \geq 0 \)).
- Keep doing this until you get to a feasible dictionary with all of the coefficients of \( z \) being \( \leq 0 \), at which point we stop and know the associated solution is optimal.
So this is a good strategy, but there’s a lot of details to worry about. What happens if the original dictionary isn’t feasible? How do we pick entering variables / exiting variables if there’s multiple choices? And how do we know this will actually land us at the optimal solution in the end? Well, we have a lot of work to do here to better understand this.

First thing: we want to set up a rule for which variables to enter and which to leave. The following is the **standard rule**:

- Choose the entering variable to be the non-basic one with the largest (positive) coefficient in \( z \). In the event of a tie, take the lowest subscript.

- Choose the exiting variable by finding which basic variable imposes the strictest bound on how far the entering variable can increase (we need to do this to keep the coefficients in the dictionary positive, i.e. to keep it feasible!). In the event of a tie, pick the one with the lowest subscript.

Alright, that’s straightforward (though not exactly what I did in my example yesterday!). You might object that sometimes you could see a better choice of variable than what this tells you, and I’d agree! This rule is picked just so we have something we can all agree on for solving problems. It isn’t really mathematically the “best” in any sense.

As a corollary to that: for the purposes of this course, I’ll be writing problems where the numbers work decently well if you follow the standard rule. If you don’t, you’re on your own with whatever awful fractions might show up in your computations!