Math 340 Lecture 31

Example started last time  Let’s say we’re managing a desk manufacturer, and there’s three styles of
desks we can have our craftspeople build, with \( x_1, x_2, x_3 \) the number of desks of each type produced each
week. We have constraints for labor (two types - the base carpentry work, and then the finishing work) plus
one for storage space in our warehouse. So our LP may look like:

- Maximize \( 12x_1 + 20x_2 + 18x_3 \) (profit margin for each desk style, in dollars), subject to:
  - \( 4x_1 + 6x_2 + 8x_3 \leq 600 \) (constraint based on carpentry labor, in hours)
  - \( x_1 + 3.5x_2 + 2x_3 \leq 300 \) (constraint based on finishing labor, in hours)
  - \( 2x_1 + 4x_2 + 3x_3 \leq 550 \) (constraint based on warehouse space, in square meters).
  - \( x_1, x_2, x_3 \geq 0 \).

Solving this (probably not so pleasant by hand!) gives us an optimal dictionary

\[
\begin{align*}
z &= 1950 - \frac{11}{2}x_4 - x_5 - 6x_3 \\
x_1 &= \frac{25}{7} - \frac{1}{19}x_4 + \frac{1}{7}x_5 - 2x_3 \\
x_2 &= \frac{75}{11} + \frac{3}{10}x_4 - \frac{3}{11}x_5 \\
x_6 &= 175 + \frac{2}{7}x_4 + \frac{1}{7}x_5 + x_3
\end{align*}
\]

So we produce \( 75/2 \) desks of type 1 and 75 of type 2 per week for maximum profit of \$1950. In preparation
for all of the sensitivity analysis we should write out all of our matrices and things:

\[
\bar{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_6 \end{bmatrix}, \quad \bar{x}_N = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}, \quad \bar{b}^* = \begin{bmatrix} 75/2 \\ 75 \\ 175 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 600 \\ 300 \\ 500 \end{bmatrix}
\]

\[
A_B = \begin{bmatrix} 4 & 6 & 0 \\ 1 & \frac{7}{2} & 0 \\ 2 & 4 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} 8 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}, \quad \bar{c}_B = \begin{bmatrix} 12 \\ 20 \end{bmatrix}, \quad \bar{c}_N = \begin{bmatrix} 18 \\ 0 \end{bmatrix}
\]

So now we can ask a bunch of questions about what happens if we change the parameters.

**Question 3: How much can we decrease the constraint for storage?** Looking at our optimal
solution, we see that the constraint for storage has a bunch of slack. In a real-world situation we might try
to decrease the unused storage space (maybe use it for something else). How much can we decrease it before
we run into a problem? One way to analyze this is to replace it with a parameter:

\[
\bar{b} = \begin{bmatrix} 600 \\ 300 \\ b_3 \end{bmatrix}
\]

We then solve \( A_B \bar{b}^* = \bar{b} \) and find

\[
\bar{b}^* = \begin{bmatrix} 75/2 \\ 75 \\ b_3 - 375 \end{bmatrix}
\]

So this stays feasible as long as \( b_3 \geq 375 \). Actually, we could pretty much have read this off from our final
dictionary - the slack variable \( x_6 \) for the constraint has an optimal value of 175, so we could certainly decrease
our initial value of 550 by as much as 175 before we get rid of all of the slack.
Question 4: What happens if the constraint for storage decreases even farther? Let’s say $b_3$ goes down below the limit of 375 above:

$$\tilde{b} = \begin{bmatrix} 600 \\ 300 \\ 350 \end{bmatrix}.$$  

Then our dictionary is no longer optimal, because it’s no longer feasible! It looks like

$$
\begin{align*}
 z &= 1950 -11x_4 -x_5 -6x_3 \\
 x_1 &= \frac{12}{2} -\frac{11}{10}x_4 +\frac{1}{2}x_5 -2x_3 \\
 x_2 &= 75 +\frac{5}{8}x_4 -\frac{7}{8}x_5 \\
 x_6 &= -25 +\frac{1}{8}x_4 +\frac{1}{2}x_5 +x_3
\end{align*}
$$

But now we know how to deal with this, because it’s a dual-feasible dictionary; we just pivot according to the dual simplex method. If we do that by working with the primal dictionary directly, we see we pick $x_6$ as the leaving variable (only one with a negative constant). To choose the entering variable we need to look at the ratios of each objective function coefficient by the corresponding (positive) coefficients in $x_6$: for $x_4$ this is $(-11/4)/(3/8) = -22/3$, for $x_5$ its $(-1)/(1/2) = -2$, and for $x_3$ it’s $(-6)/1 = -6$. The smallest-absolute-value one is $x_5$, so we choose that as our entering variable. The resulting dictionary is

$$
\begin{align*}
 z &= 1900 -2x_4 -2x_6 -4x_3 \\
 x_1 &= 75 -1x_4 +\frac{1}{2}x_6 -\frac{1}{2}x_3 \\
 x_2 &= 50 +\frac{1}{4}x_4 -x_6 +x_3 \\
 x_5 &= 50 -\frac{3}{4}x_4 +2x_6 -2x_3
\end{align*}
$$

So that’s optimal again, this time with $x_1, x_2, x_5$ as basic variables (so we produce 75 desks of type 1, 50 of type 2, and 0 of type 3, and end up with slack in our constraint for finishing work).

Question 5: What happens if we increase profit on Desk 3? So far, all of our solutions have involved $x_3 = 0$, i.e. desk type 3 has never been worth producing. How much would the net profit on it have to increase for it to start being worthwhile? Or, if we change its profit to a variable

$$\tilde{c} = \begin{bmatrix} 12 \\ 20 \\ c_3 \end{bmatrix},$$

for what range of $c_3$ does our current solution having $x_3 = 0$ stay optimal? (We clearly need to go beyond this range to have a chance of production of $x_3$ to be worthwhile!)

To solve this we just need to do our usual thing of writing out the coefficients of the objective function when $c_3$ is a variable: the formula is $\tilde{c}_N^T - \tilde{c}_B^T A_B^{-1} A_N = \tilde{c}_N^T - \tilde{y}^T A_N$. We haven’t directly solved for $\tilde{y}$ yet but it’s closely related to the marginal values and is in fact

$$\tilde{y} = \begin{bmatrix} 11/4 \\ 0 \end{bmatrix}.$$  

Plugging this in we find

$$\tilde{c}_N^T - \tilde{y}^T A_N = \begin{bmatrix} b_3 - 24 & -11/4 & -1 \end{bmatrix}.$$  

So as long as $b_3 \leq 24$, our original solution remains optimal and it is not profitable to produce any of Desk 3.

Actually, we could have concluded this with less computation by looking at our final dictionary. We knew originally $\tilde{c}_N^T = \begin{bmatrix} 18 & 0 & 0 \end{bmatrix}$ and for that our coefficients were $\begin{bmatrix} -6 & -11/4 & -1 \end{bmatrix}$; from this we can reverse-engineer that $\tilde{y}^T A_N = \begin{bmatrix} -24 & -11/4 & -1 \end{bmatrix}$ and proceed from there (or just see directly that increasing $b_3$ will correspondingly increase the coefficient $-6$ of $x_3$, so we need to get up to 24 to “break even” on profitability).
Question 6: How does the solution change if Desk 3 is made profitable? So let’s say we can increase the profit margin on Desk 3 to $25, one way or another, to break out of the range determined in the previous problem. What’s our new optimal solution? Once again we go back to our dictionary with $x_B$ and modify it for our new objective function:

$$
\begin{align*}
  z &= 1950 - \frac{11}{3}x_4 - x_5 + x_3 \\
  x_1 &= \frac{10}{7} - \frac{11}{9}x_4 + \frac{2}{3}x_5 - \frac{2}{5}x_3 \\
  x_2 &= \frac{75}{4} + \frac{5}{8}x_4 - \frac{1}{3}x_5 \\
  x_6 &= 175 + \frac{5}{2}x_4 + \frac{2}{9}x_5 + x_3 \\
\end{align*}
$$

The simplex method tells us to let $x_3$ enter and $x_1$ leave, which we can do, though it makes things even uglier:

$$
\begin{align*}
  z &= \frac{7875}{4} - \frac{95}{112}x_4 - \frac{5}{8}x_5 - \frac{1}{6}x_1 \\
  x_3 &= \frac{10}{4} + \frac{17}{112}x_4 + \frac{5}{8}x_5 - \frac{1}{6}x_1 \\
  x_2 &= \frac{75}{4} + \frac{5}{8}x_4 - \frac{1}{3}x_5 \\
  x_6 &= \frac{775}{4} + \frac{5}{32}x_4 + \frac{7}{8}x_5 - \frac{1}{2}x_1 \\
\end{align*}
$$

Note $7875/4 = 1968.75$, so we’ve moderately increased our original profit of 1950 by switching our production from $(x_1, x_2, x_3) = (75/2, 75, 0)$ to $(x_1, x_2, x_3) = (0, 75, 75/4)$. 

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