Sensitivity analysis, continued. Let’s keep going with the example from last time:

- Maximize \(2x_1 + x_2 + 2x_3\), subject to:
  - \(x_1 + x_3 \leq 1\)
  - \(x_2 + 2x_3 \leq 2\)
  - \(x_1 + 2x_2 + 3x_3 \leq 3\)
  - \(x_1, x_2, x_3 \geq 0\).

In matrix form:

\[
A_{\text{aug}} = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 1 & 0 \\
1 & 2 & 3 & 0 & 0 & 1
\end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{c}_{\text{aug}} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

optimal dictionary is

\[
\vec{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{x}_N = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}, \quad \vec{b}^* = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

and corresponds to an optimum value of \(z = 3\) occurring when \(x_1 = x_2 = 1\) and \(x_3 = 0\) (and \(x_4 = x_6 = 0\) and \(x_5 = 1\)). We have

\[
A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}, \quad A_N = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad \vec{c}_B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{c}_N = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.
\]

For reference the formula for the dictionary is

\[
\vec{x}_B = A_B^{-1}\vec{b} - A_N^{-1}A_N\vec{x}_N = \vec{c}_B + (\vec{c}_N - A_B^{-1}A_N\vec{x}_N)\vec{x}_N.
\]

Last time we talked about how to analyze changes to the objective function (remember we were only changing one parameter at a time) - we generally found that there would be a range in which the same dictionary stayed optimal. Now let’s consider changing some other things, like an entry in the matrix \(A\).

What if the coefficient of \(x_3\) in \(x_1 + 2x_2 + 3x_3 \leq 3\) changed? Then we’d have

\[
A_{\text{aug}} = \begin{bmatrix}
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 2 & 0 & 1 & 0 \\
1 & 2 & 3 & 0 & 0 & 1 \\
a_{33} & 0 & 0 & 1 \\
\end{bmatrix},
\]

with \(a_{33}\) a parameter. This \(a_{33}\) only shows up in \(A_N\) so we get

\[
\begin{bmatrix}
3/2 & 0 & 1/2 \\
2 & 0 & 0 \\
\end{bmatrix} + \begin{bmatrix}
1/2 & 0 \\
0 & 1/2 \\
\end{bmatrix} \begin{bmatrix}
1 & 1 & 0 \\
2 & 0 & 0 \\
a_{33} & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
x_3 \\
x_4 \\
x_6 \\
\end{bmatrix},
\]

and thus \(z = 3 + (1/2 - a_{33})x_3 - 3/2x_4 - 1/2x_6\). So in this case we remain optimal as long as \(a_{33} \geq 1\), and if it got lower than that then \(x_3\) would start to become worthwhile to produce.
What if we changed the coefficient 2 of \( x_2 \) in \( x_1 + 2x_2 + 3x_3 \leq 3? \) Well, then things get a bit more complicated, because this coefficient is in \( A_B \). So it will change \( y \) and affect the objective value and objective function, but also change the row of constants \( A_B^\top \mathbf{b} \). This didn’t happen from any of the other changes we’d made thus far. Nonetheless we can still analyze where changes to \( a_{32} \) would leave us. We need to find our new vector of constants \( A_B^\top \mathbf{b} = \mathbf{b}^* \), but we can do this as usual, by solving \( A_B \mathbf{b}^* = \mathbf{b} \):

\[
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 2 \\
1 & a_{32} & 0 & 3
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & a_{32} & 0 & 2 \\
0 & 0 & a_{32} & 1
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 2 \\
0 & 0 & 1 & 1
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2/a_{32} \\
0 & 0 & 1 & 2 - 2/a_{32}
\end{pmatrix}.
\]

So our new constant vector is

\[
\mathbf{b}^* = \begin{bmatrix} 1 \\
2/a_{32} \\
2 - 2/a_{32} \end{bmatrix},
\]

i.e., trying to set \( x_3, x_4, x_6 \) to be nonbasic variables forces the basic variables to take values \( x_1 = 1, x_2 = 2/a_{32} \), and \( x_5 = 2 - 2/a_{32} \). So when is this feasible? Need \( a_{32} > 0 \) for \( x_2 \) to be positive, and then need \( a_{32} \geq 1 \) for \( x_5 \) to be nonnegative. (And we needed \( a_{32} \neq 0 \) in the first place for our division to even make any sense - if \( a_{32} = 0 \) then \( A_B \) fails to be invertible which means \( \mathbf{x}_B \) wouldn’t be a valid thing we could pivot to for that modified problem!)

So alright, as long as \( a_{32} \geq 1 \), we keep feasibility of our original optimal dictionary even with this change. What about optimality? Again we need to solve \( A_B^\top \mathbf{y} = \mathbf{c} \_B \):

\[
\begin{pmatrix}
1 & 0 & 1 & 2 \\
0 & 1 & a_{32} & 1 \\
0 & 1 & 0 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 2 - 1/a_{32} \\
0 & 0 & a_{32} & 0 \\
0 & 0 & 1 & 1/a_{32}
\end{pmatrix}
\Rightarrow
\mathbf{y} = \begin{bmatrix}
2 - 1/a_{32} \\
0 \\
1/a_{32}
\end{bmatrix}.
\]

Then we go back to the usual formula for the objective function:

\[
\begin{bmatrix}
2 - 1/a_{32} & 0 & 1/a_{32}
\end{bmatrix}
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
+ \begin{bmatrix}
2 & 0 & 0
\end{bmatrix}
- \begin{bmatrix}
2 - 1/a_{32} & 0 & 1/a_{32}
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 \\
2 & 0 & 0 \\
3 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_3 \\
x_4 \\
x_6
\end{bmatrix},
\]

which equals

\[
\left(2 + \frac{2}{a_{32}}\right) - \frac{2}{a_{32}}x_3 + \left(-2 + \frac{1}{a_{32}}\right)x_4 - \frac{1}{a_{32}}x_6.
\]

So when is this optimal? Well, we know we’re restricted to \( a_{32} \geq 1 \) anyway, and on that range the answer is “always”? The coefficients of \(-2/a_{32} \) and \(-1/a_{32} \) will always be negative, and \(-2 + 1/a_{32} \) will be at most \(-1 \). So we conclude:

- As long as the entry \( a_{32} \) is \( \geq 1 \), our same choice of basic variables \( \mathbf{x}_B \) will lead to an optimal dictionary with \( x_1 = 1, x_2 = 2/a_{32}, x_3 = 0 \), and objective value \( 2 + 2/a_{32} \).

In this case if \( a_{32} \) dips below 1, the dictionary becomes infeasible, though it will still stay dual-feasible if \( a_{32} \geq 1/2 \), so we could proceed using the dual simplex method to fix things. If \( a_{32} \) gets below that (not to mention if it gets negative) then it’s neither feasible nor dual-feasible and at that point it’s hard to get much information from our optimal solution in the case \( a_{32} = 2 \). But intuitively that should make sense - if we mess around with the coefficient in that constraint (which is some sort of unit cost for \( x_2 \)) just a little things are okay, but if you drop it really low then a significantly different solution might happen.

So we’ve done an example of changing the objective function, and the coefficient matrix \( A \). Of course we can also change the vector of constants \( \mathbf{b} \), and hopefully the idea of what we’re doing looks familiar now. Also this is the one example we’ve sort of talked about before - the marginal values theorem talked about how the optimal dual solution would give us shadow prices for how much it was worth to us to change the constants. There we just said that things would work for “small changes” but now we have a way to figure out quantitatively what the range is.
Let’s say we want to study how the constant \( x_1 + x_3 \leq 1 \) in the first constraint affects the solution. As usual we change this to \( x_1 + x_3 \leq b_1 \) and study things from there. Looking at the formulas for dictionaries what actually changes is the constants in the final dictionary, and the objective function, so we just need to compute those. The constants are \( \mathbf{b}^* = A_B^{-1} \mathbf{b} \), so we solve \( A_B \mathbf{b}^* = \mathbf{b} \):

\[
\begin{bmatrix}
1 & 0 & 0 & b_1 \\
0 & 1 & 1 & 2 \\
1 & 2 & 0 & 3
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & b_1 \\
0 & 2 & 0 & 3-b_1 \\
0 & 1 & 1 & 2
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & \frac{3-b_1}{2} \\
0 & 1 & 1 & \frac{3-b_1}{2} \\
0 & 0 & 1 & \frac{b_1}{2}
\end{bmatrix}
\]

and thus

\[
\mathbf{b}^* = \begin{bmatrix}
b_1 \\
\frac{3-b_1}{2} \\
\frac{b_1}{2}
\end{bmatrix}.
\]

So our dictionary remains feasible (and thus optimal, since the objective function hasn’t changed!) as long as \( b_1 \geq 0 \) (to guarantee the first and third entry are \( \geq 0 \)) and \( b_1 \leq 3 \) (to guarantee the second is \( \geq 0 \)), so our range of optimality is \( 0 \leq b_1 \leq 3 \). Outside of that range (say, if we bump \( b_1 \) up to 4) the dictionary becomes infeasible, but it remains dual-feasible so we could use the dual simplex method from this dictionary to find the new optimum.

In the range \( 0 \leq b_1 \leq 3 \) the optimum value is

\[
\mathbf{c}^T_B A_B^{-1} \mathbf{b} = \mathbf{c}^T_B \mathbf{b}^* = \begin{bmatrix} 2 & 1 & 0 \end{bmatrix} \begin{bmatrix}
b_1 \\
\frac{3-b_1}{2} \\
\frac{b_1}{2}
\end{bmatrix} = 2b_1 + \frac{3-b_1}{2} = \frac{3}{2} b_1 + \frac{3}{2}
\]

This lines up with the marginal values theorem - when we solved \( \mathbf{y} \) we found \( y_1 = \frac{3}{2} \), which meant that the marginal price for the first constraint was \( \frac{3}{2} \). And sure enough, increasing \( b_1 \) by 1 will increase our optimum value by \( \frac{3}{2} \)!