Math 340 Lecture 22

An extended example of the revised simplex method. Let’s go through the revised simplex method for the following example:

- Maximize \( 3x_1 + x_2 + 2x_3 + 5x_4 \) subject to
  - \( 3x_1 + x_1 + 4x_3 + 7x_4 \leq 15 \)
  - \( 2x_1 + x_2 + x_3 + 5x_4 \leq 10 \)

To give this some concrete meaning, let’s say this is a factory optimization problem - \( x_1, x_2, x_3, x_4 \) represent quantities of four goods that could be made in a day, the coefficients of the objective function represents profit per object. The first constraint represents hours of labor available vs. the hours required for production of a unit of each type of good, while the second represents how much machinery the factory has available for use vs. how much of it needs to be dedicated per unit of each type of product.

So alright, since this is a relatively small system (few variables and few constraints) it would be easy to solve this via the original simplex method with dictionaries, but the point is I want to demonstrate how you’d solve something with the revised simplex method and not keeping track of the full dictionary (which would likely be more practical if this was a large-scale real-world problem with 4000 variables and 2000 constraints rather than 4 and 2).

Writing out the matrices, and pivot 1. The revised simplex method relies on matrix notation, so we need to write those out to work with as we go along. We have

\[
A_{aug} = [A | I] = \begin{bmatrix} 3 & 1 & 4 & 7 & 1 & 0 \\ 2 & 1 & 1 & 5 & 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}, \quad \vec{c}_{aug} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 5 \\ 0 \\ 0 \end{bmatrix}.
\]

How do we get started with the first step of the revised simplex method? Well, we need to have our \( \vec{x}_B \) and \( \vec{x}_N \) (and the associated vector of constants \( \vec{b}^* \)), but this turns out to be the obvious thing: we take our basic variables to be the slack ones and the nonbasic variables to be the decision ones, as in a normal starting dictionary, and then the constants of the dictionary are just \( \vec{b} \):

\[
\vec{x}_B = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix}, \quad \vec{x}_N = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}.
\]

Now we’re ready to get started with our pivoting. We might be tempted to dive in to the formulas and the whole method, but for the very first pivot things are easy: we already have the entire dictionary written right in front of us! Even if we went through the formulas we’d find \( A_B \) is the identity and the vector of coefficients for the objective function is just our original \( \vec{c} \). So we can just bypass doing that for this very first pivot, and read off that we want \( x_4 \) to be our entering variable. Similarly we can read off that \( x_6 \) should be the exiting variable - if we take the constants minus \( A_4 \) (the coefficients for \( x_4 \) in the dictionary) we have

\[
\begin{bmatrix} 15 - 7t \\ 10 - 5t \end{bmatrix} = \begin{bmatrix} 15 - 7x_4 \\ 10 - 5x_4 \end{bmatrix}
\]

and at \( x_4 = 2 \) the value of \( x_6 \) hits zero.

Just like that we’re done the first pivot, and we can write out the \( \vec{x}_B \) and \( \vec{x}_N \) for the next step. But according to the revised simplex method as I wrote it out, we want to also find the new vector of constants
\( \mathbf{\hat{b}}^* \), i.e. the constants of our new basic variables \( x_5 \) and \( x_4 \) in our new dictionary. What are they? For \( x_5 \) it’s the value \( 15 - 7 \cdot 2 = 1 \) that was left when we increased \( x_4 \) to the final value it could go above. For \( x_4 \) it’s simply the value of 2 we obtained! So we have

\[
\begin{align*}
\bar{x}_B &= \begin{bmatrix} x_5 \\ x_4 \end{bmatrix}, \\
\bar{x}_N &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_6 \end{bmatrix}, \\
\mathbf{\hat{b}} &= \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\end{align*}
\]

Note that here I’ve chosen to swap the places of \( x_4 \) and \( x_6 \) directly. I could have just as well re-ordered \( \bar{x}_B \) to have \( x_4 \) before \( x_5 \), and it wouldn’t make anything that followed different, as long as I took one order of variables in \( \bar{x}_B \) and stuck with it throughout the entire next step.

**Pivot 2.** Now we get to the first pivot where we have to do a bit of work. Let’s start by writing out the matrices and vectors we need:

\[
\begin{align*}
A_B &= \begin{bmatrix} 1 & 7 \\ 0 & 5 \end{bmatrix}, & A_N &= \begin{bmatrix} 3 & 1 & 4 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix}, & \mathbf{\hat{c}}_B &= \begin{bmatrix} 0 \\ 5 \end{bmatrix}, & \mathbf{\hat{c}}_N &= \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix}.
\end{align*}
\]

Step 1 of the revised simplex method is to compute \( \mathbf{\bar{y}} = (A_B^T)^{-1}\mathbf{\hat{c}}_B \) via solving \( A_B^T \mathbf{\bar{y}} = \mathbf{\hat{c}}_B \):

\[
\begin{align*}
\begin{bmatrix} 1 & 0 & 0 \\ 7 & 5 & 5 \end{bmatrix} & \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{\bar{y}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{align*}
\]

We then compute the coefficients of the objective function with our usual formula

\[
\mathbf{\hat{e}}_N^T - \mathbf{\hat{e}}_B A_B^{-1} A_N = \mathbf{\hat{e}}_N^T - \mathbf{\bar{y}}^T A_N = \begin{bmatrix} 3 & 1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \end{bmatrix}.
\]

From this we see the first and third columns are the best choice for entering variables, i.e. \( x_1 \) and \( x_3 \) - by our usual tiebreaker of smaller index we choose \( x_1 \) to enter.

Now, we need to find the leaving variable. Once again we need to find \( \mathbf{\bar{d}} = A_B^{-1} \hat{A}_1 \) (where \( \hat{A}_1 \) is the column of \( A_N \) or \( A_{aug} \) corresponding to the variable \( x_1 \)) by way of solving \( A_B \mathbf{\bar{d}} = \hat{A}_1 \):

\[
\begin{align*}
\begin{bmatrix} 1 & 7 & 3 \\ 0 & 5 & 2 \end{bmatrix} & \sim \begin{bmatrix} 1 & 7 & 3 \\ 0 & 1 & .4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & .2 \\ 0 & 1 & .4 \end{bmatrix} \quad \Rightarrow \quad \mathbf{\bar{d}} = \begin{bmatrix} .2 \\ .4 \end{bmatrix}.
\end{align*}
\]

(I’ll be using decimals here because everything turns out to be fractions of \( 1/5 \) which makes it easy). We next look at

\[
\mathbf{\hat{b}}^* - x_1 \mathbf{\bar{d}} = \begin{bmatrix} 1 - .2x_1 \\ 2 - .4x_1 \end{bmatrix}.
\]

Here we find that if we increase \( x_1 \) to 5 then both terms become 0. So we have a tie for exiting variables \( x_5 \) and \( x_4 \), and we choose \( x_4 \) to leave because it has a lower index (even though it’s the bottom row and \( x_5 \) is the top in our ordering). Then we have

\[
\begin{align*}
\hat{x}_B &= \begin{bmatrix} x_5 \\ x_1 \end{bmatrix}, \\
\hat{x}_N &= \begin{bmatrix} x_4 \\ x_2 \\ x_3 \\ x_6 \end{bmatrix}, \\
\mathbf{\hat{b}}^* &= \begin{bmatrix} 0 \\ 5 \end{bmatrix}.
\end{align*}
\]

Here, the constant term for \( x_5 \) is the value of 5 we found above, and the constant for \( x_4 \) is 0 = 1 − .2 · 5.
**Interpretation of this.** Before proceeding to the next pivot, let’s talk about our interpretation of the variable \( \mathbf{y} \) we found. Last time we talked about \( \mathbf{y} \) being obtained in a way similarly to how we use complementary slackness. Another interpretation is in terms of economics, sort of like the marginal values theorem we talked about before. Here the values we’ve calculated of

\[
\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

are interpreted as *shadow prices* for our two constraints, based on the dictionary we were working from. In our case this corresponded to a feasible solution of \( x_1 = x_2 = x_3 = 0 \) and \( x_4 = 2 \), producing giving a total of \( 3 \cdot 0 + 1 \cdot 0 + 2 \cdot 0 + 5 \cdot 2 = \$10 \); this dictionary amounts to valuing each hour of labor at \$1\) (because we use 10 machines to make \$10 of profit) and labor at \$0\) (because we still have some unused labor). Numerically this makes sense since we found \( y_1, y_2 \) by solving the system of equations \( y_1 = 0 \) and \( 7y_1 + 5y_2 = 0 \), amounting to us setting our shadow prices by choosing to value our basic variables \( x_4 \) (“production of good 4”) and \( x_5 \) (“leaving labor idle”). If this solution were optimal, it would make sense for the marginal value of labor to be \$0\), since we have an abundance of it already so paying for even more wouldn’t help us improve production.

Now, what does the computation of \( \mathbf{c}_B^T - \mathbf{y}^T \mathbf{A}_N \) mean? Transposing, we’re comparing \( \mathbf{c}_B \) to \( A_N^T \mathbf{y} \). Each value of \( \mathbf{c}_B \) corresponds to the per-unit profit of increasing the corresponding variables \( (x_1, x_2, x_3, \) or \( x_6 \) respectively; in real terms these amount to adding in production of goods 1, 2, 3, and 3, to leaving labor idle\). The corresponding value of \( A_N^T \mathbf{y} \) gives the value of our current use of the resources involved in increasing \( x_1, x_2, x_3, x_6 \), according to how we value labor and machines per our current shadow prices. So increasing one unit of \( x_1 \) would require us to use 3 hours of labor and 2 machines, which is valued at \( 3 \cdot \$0 + 2 \cdot \$1 = \$2 \) per our shadow prices. Since producing a unit of \( x_1 \) profits \$3\) it makes sense to increase production of \( x_1 \) vs. our current choices getting \$2 from the same resources, i.e. to choose it as an entering variable!

**Pivot three.** Again we start by writing out our matrices and vectors:

\[
\mathbf{A}_B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{A}_N = \begin{bmatrix} 7 & 1 & 4 & 0 \\ 5 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} 5 \\ 1 \\ 2 \\ 0 \end{bmatrix}.
\]

We solve \( \mathbf{A}_B^T \mathbf{y} = \mathbf{c}_B \):

\[
\begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1.5 \end{bmatrix} \Rightarrow \mathbf{y} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}.
\]

Then we compute

\[
\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_N = \mathbf{c}_N^T - \mathbf{y}^T \mathbf{A}_N = \begin{bmatrix} 5 & 1 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1.5 \end{bmatrix} \begin{bmatrix} 7 & 1 & 4 & 0 \\ 5 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2.5 & -0.5 & 0.5 & -1.5 \end{bmatrix}.
\]

So we choose \( x_3 \) (corresponding to our third column) to enter. We then pick the exiting variable as usual, first solving \( \mathbf{A}_B \mathbf{d} = \mathbf{A}_3 \):

\[
\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & .5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2.5 \\ 0 & 1 & .5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2.5 \\ .5 \end{bmatrix}.
\]

We then look at

\[
\bar{b}^* - x_3 \mathbf{d} = \begin{bmatrix} 0 - 2.5x_3 \\ 5 - .5x_3 \end{bmatrix}.
\]
Here we have a degenerate pivot, where we can only take $x_3 = 0$ before having to choose $x_5$ to exit. Doing this we find that $x_1$ keeps its value of 5 too, so we have:

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_1 \end{bmatrix} \quad \mathbf{x}_N = \begin{bmatrix} x_4 \\ x_2 \\ x_5 \\ x_6 \end{bmatrix} \quad \tilde{\mathbf{b}}^* = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$  

**Attempted pivot four.** So we’ve reached a feasible solution of $x_1 = 5$ and $x_2 = x_4 = x_5 = 0$, which produces a profit of $15 and uses all of both of our labor and machinery. Is this optimal? We need to check our coefficients once again.

$$A_B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \quad A_N = \begin{bmatrix} 7 & 1 & 1 & 0 \\ 5 & 1 & 0 & 1 \end{bmatrix} \quad \mathbf{c}_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \mathbf{c}_N = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$  

We compute

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 4 & 1 & 2 \\ -5 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 4 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & .2 \\ 0 & 1 & 1.2 \end{bmatrix} \quad \Rightarrow \quad \mathbf{y} = \begin{bmatrix} .2 \\ 1.2 \end{bmatrix},$$  

and then

$$\mathbf{c}_N^T - \mathbf{y}^T A_N = \begin{bmatrix} 5 & 1 & 0 & 0 \end{bmatrix} - [.2 & 1.2] \begin{bmatrix} 7 & 1 & 1 & 0 \\ 5 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2.4 & -4 & -2 & -1.2 \end{bmatrix}. $$

So all of our coefficients are negative and we’re done!