Math 340 Lecture 20

The actual revised simplex method, version 1. Last time I gave a preliminary outline of the revised simplex method, which turned out to be not so good since it involved $A_B^{-1}$’s all over the place. So how can we make the revised simplex method actually work reasonably efficiently, i.e. not spend a massive amount of time computing $A_B^{-1}$ at each step?

- Approach 1: Notice that we’re not actually ever using $A_B^{-1}$ by itself, just $A_B^{-1}$ times vectors. So use linear algebra to find these products more efficiently.
- Approach 2: Reduce computations by keeping track of $A_B^{-1}$ from step to step - if we have $A_B^{-1}$ at the beginning of one iteration of the simplex method, we can try to use it to get the new $A_B^{-1}$ for the next iteration without recomputing the whole thing.

We’ll actually give two different ways to do the simplex method, one based primarily on each approach. We’ll start with approach 1, thinking through how we can compute each piece of data in our above summary more efficiently.

0. Start our iteration with $\vec{x}_B$ and $\vec{x}_N$ given, and $\vec{b}^* = A_B^{-1}\vec{b}$ given to us from the end of the previous iteration.

1. Obtain $\vec{y} = (A_B^{-1})^T \vec{c}_B$ by solving the system of equations $A_B^T \vec{y} = \vec{c}_B$ (using row-reduction as usual). Then compute $\vec{c}_N^T - \vec{c}_B A_B^{-1} A_N = \vec{c}_N^T - \vec{y}^T A_N$.

2. The largest coefficient in the row vector we just computed corresponds to a variable $x_j$ in $\vec{x}_N$ that we choose as our entering variable. (As usual in the event of a tie we pick the $x_j$ with a smaller index).

3. Obtain $\vec{d} = A_B^{-1} \vec{A}_j$ by solving the system of equations $A_B \vec{d} = \vec{A}_j$.

4. For each entry of $(A_B^{-1}\vec{b}) - t(A_B^{-1} \vec{A}_j) = \vec{b}^* - t\vec{d}$, find what value of $t$ makes it zero (i.e. just divide the entry from the first vector by the corresponding entry from the second).

5. Take the smallest nonnegative value from the list above, and take the exiting variable $x_i$ to correspond to the row it was from. (Again pick the smaller index in the event of a tie). Also, compute the vector of constants $\vec{b}^*$ for the new dictionary after pivoting: it will equal $\vec{b}^* - t\vec{d}$ for the smallest $t$ we just picked, except the constant corresponding to $x_i$ will be replaced by the new constant for $x_j$ (which is the constant in $\vec{b}^*$ corresponding to $x_i$ divided by the coefficient in $\vec{d}$ corresponding to $x_i$).

An example: Let’s go back to one of the first LPs we looked at.

- Maximize $6x_1 + 8x_2 + 5x_3 + 9x_4$ subject to:
  - $2x_1 + x_2 + x_3 + 3x_4 \leq 5$
  - $x_1 + 3x_2 + x_3 + 2x_4 \leq 3$
  - $x_1, x_2, x_3, x_4 \geq 0$.

Let’s say after a few steps of the simplex method we’re currently at

$$
\vec{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} \quad \vec{x}_N = \begin{bmatrix} x_2 \\ x_3 \\ x_5 \\ x_6 \end{bmatrix} \quad \vec{b}^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
$$

From this we can read off

$$
A_{aug} = \begin{bmatrix} 2 & 1 & 1 & 1 & 3 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 & 1 \end{bmatrix} \implies A_B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \quad A_N = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}
$$

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\[ \vec{c}_{aug} = \begin{bmatrix} 6 \\ 8 \\ 5 \\ 9 \\ 0 \\ 0 \end{bmatrix} \implies \vec{c}_B = \begin{bmatrix} 6 \\ 9 \end{bmatrix} \quad \vec{c}_N = \begin{bmatrix} 8 \\ 5 \\ 0 \\ 0 \end{bmatrix}. \]

(In this case it would be easy to find \( A_B \) since it’s only \( 2 \times 2 \), but I’ll avoid doing that so we can see how we’d do things for larger matrices). For step 1 we need to solve \( A_B \bar{y} = \vec{c}_B \), so we go back to linear algebra and remember we solve this via row-reducing the augmented matrix:

\[
\begin{bmatrix} A_B & | & \vec{c}_B \end{bmatrix} = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 2 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 1/2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \implies \bar{y} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},
\]

and we read off the solution. Then the coefficients of the objective function in this dictionary are

\[
\vec{c}_N^T - \bar{y}^T A_N = \begin{bmatrix} 8 & 5 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -3 & 0 \end{bmatrix},
\]

and we see the largest one is 5 corresponding to the nonbasic variable \( x_2 \), which is thus our entering variable.

Now we move onto step 3 where we find \( A_B^{-1} \tilde{\vec{A}}_2 \) for \( \tilde{\vec{A}}_2 \) the column of \( A \) corresponding to our entering variable. Again, we do this by solving the system of equations \( A_B \tilde{\vec{d}} = \tilde{\vec{A}}_j \):

\[
\begin{bmatrix} A_B & | & \tilde{\vec{A}}_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \end{bmatrix} \implies \tilde{\vec{d}} = \begin{bmatrix} -7 \\ 5 \end{bmatrix}.
\]

So: we then look at

\[
\vec{b}^* - t\tilde{\vec{d}} = \begin{bmatrix} 1 + 7t \\ 1 - 5t \end{bmatrix}.
\]

Here the first row will always be positive for positive \( t \), while the second row will become zero at \( t = 1/5 \). So the second row (corresponding to the basic variable \( x_4 \)) imposes the tightest constraint, and thus \( x_4 \) is our exiting variable. For this value of \( t \) we have

\[
\vec{b}^* - t\tilde{\vec{d}} = \begin{bmatrix} 12/5 \\ 0 \end{bmatrix},
\]

and the new \( \vec{b}^* \) for the new dictionary will be this but with the 0 in the row for the exiting variable replaced by 1/5 (because this is the new coefficient for the entering variable \( x_2 \)). So then we’re done our iteration and ready to go on to the next one (which I won’t do here).