Math 340 Lecture 2

We had a couple of examples last class, so now we have an idea of what a linear program is. General setup: going back to linear algebra, a linear function on \( n \) variables (usually we’ll denote them \( x_1, \ldots, x_n \)) is any function of the form

\[
f(x_1, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n = \sum_{k=0}^{n} a_kx_k
\]

for some real numbers \( a_1, \ldots, a_n \) (so \( f_1(x_1, x_2, x_3) = 2x_1 + x_2 - x_3 \) and \( f_2(x_1, x_2) = 4.5x_2 \) are two examples in three variables - in the latter example \( x_1 \) is still a variable but just has coefficient 0). A linear constraint on our variables is an expression that looks like one of the following things, for a linear function \( f(x_1, \ldots, x_n) \) and a constant \( c \):

\[
\begin{align*}
f(x_1, \ldots, x_n) &\geq c, \\
f(x_1, \ldots, x_n) &\leq c, \\
f(x_1, \ldots, x_n) &= c
\end{align*}
\]

We do not allow strict inequalities here! (If you’re curious about why, think about what it would mean to try to do something like maximize \( x \) subject to \( x < 1 \).)

So looking back at an example from last time, the gas blending problem:

- Minimize \( 45t + 30a + 30p \), subject to the constraints:
  - \( t + a + p = 1 \)
  - \( 100t \geq 5 \)
  - \( 2t + 4.8a + 19.7p \geq 5.5 \)
  - \( 2t + 4.8a + 19.7p \leq 7 \)
  - \( 100t + 125a + 125p \geq 115 \)
  - \( t, a, p \geq 0 \).

Most of the data here is a list of linear constraints. The remaining thing is the linear function \( 45t + 30a + 30p \) that we want to minimize; this is the objective function. So:

**Definition 1.** A linear program or linear programming problem on variables \( x_1, \ldots, x_n \) consists of:

- A list of linear constraints we want the variables to satisfy.
- A linear objective function \( z = f(x_1, \ldots, x_n) \) we want to either minimize or maximize.

A point \( (x_1, \ldots, x_n) \in \mathbb{R}^n \) (an assignment of the \( n \) variables) is called a feasible solution if it satisfies all of the linear constraints. It’s called an optimal solution if it’s a feasible solution and furthermore its cost is \( \leq \) the cost of all other feasible solutions.

So our goal is to be able to find an optimal solution for any given linear programming problem. There are a few things we need to look out for, though, illustrated by the following silly examples in two variables \( x, y \):

- A linear program can have no feasible solutions at all (and is accordingly called infeasible)! Example: say I put the constraints \( x + y \geq 2 \) and \( x + y \leq 0 \). Doesn’t even matter what the objective function is.
- A linear program can have feasible solutions, but no optimal solutions. Example: constraints \( x + y \geq 5 \) and \( 2x - y \geq 3 \), and we want to maximize \( 3x \). Well, you can’t, because you can actually get values going off to infinity - if I want to get \( 3x = 3000000 \) I just need to find a feasible solution with \( x = 1000000 \), and \( (x, y) = (1000000, 0) \) satisfies our constraints (as do plenty of other points).
- A linear program can have multiple optimal solutions (but only one optimal value that the objective function takes for all of them). Example: want to maximize $x + y$ subject to $x + y \leq 5$, $x \geq 0$, and $y \geq -7$. Well, obviously the maximum value is 5, but you can get at that by $(x, y) = (5, 0)$ or $(0, 5)$ or $(2, 3)$ or $(7, -2)$, etc.

These examples are easy to see what’s happening with because of how they’re written and because there’s only two variables, but with more complicated linear programs in more variables it can be hard to tell just by looking whether something goes “wrong” like this. (There’s nothing really “wrong” with the third example, though, since having more than one optimal choice can happen even in real-world problems).

**Standard form for linear programs.** The general definition of linear programs above makes sense, but is a bit unwieldy to work with algorithmically, in part because there’s so many “or’s” (minimize or maximize, $\leq$ or $\geq$ or $=$. For the simplex method we’ll be focusing on linear programs in something called standard form:

**Definition 2.** A linear program is in standard form if it’s asking you to maximize the objective function, it includes positivity constraints $x_i \geq 0$ for every variable, and all of the other constraints are of the form $f(x_1, \ldots, x_n) \leq c$.

At first this seems like a big restriction - for instance our gas blending problem definitely doesn’t fit these rules! But thinking about it a bit, that turns out to be just an artifact of how we’ve written it. The gas blending problem asks us to minimize $45t + 30a + 30p$, but that’s the same as maximizing its negative. And similarly, there’s a bunch of constraints that are written as $\geq$, but we can rewrite them as $\leq$ by putting a negative sign in front of both sides (remember multiplying by a negative number reverses inequalities). The equality constraint $t + a + p = 1$ isn’t allowed easier, but we can rewrite this as two inequalities $t + a + p \geq 1$ and $t + a + p \leq 1$ because together these inequalities sandwich $t + a + p$ and force it to equal 1 (and of course, we then need to rewrite the first one as $\leq$).

So the gas blending problem can be equivalently written in the following way, which is in standard form.

- Maximize $-45t - 30a - 30p$, subject to the constraints:
  - $t + a + p \leq 1$
  - $-t - a - p \leq -1$
  - $-100t \leq -5$
  - $-2t - 4.8a - 19.7p \leq -5.5$
  - $2t + 4.8a + 19.7p \leq 7$
  - $-100t - 125a - 125p \leq -115$
  - $t, a, p \geq 0$.

So: for whatever linear program you’re starting with, it’s pretty easy to just rewrite the objective function so it’s always being maximized, and to rewrite the constraints to all be $\leq$. If it started with positivity constraints on all of the variables, it will then be in standard form and we’ll be able to feed it into the simplex method and solve it.

What if we didn’t start with a positivity constraint on the variable $x$, say? This isn’t so common but can still happen in practice (say $x$ represents a monetary balance, and it could either go positive by saving money or negative by taking out loans). How would we convert this into standard form? Well, we can use a little trick: replace $x$ by $y - z$ everywhere for two variables $y, z$ which we can constrain by $y, z \geq 0$. This works because any value of $x$ can be represented by some choice (actually, many possible choices) of $y, z \geq 0$ with $x = y - z$, the most obvious being $(y, z) = (x, 0)$ if $x \geq 0$ and $(y, z) = (0, |x|)$ if $x \leq 0$. In any case if we solve the new system with $y, z$ we can go back and find what the original variable $x$ would be.

If $x$ is allowed to be negative but has a lower bound (say $x \geq -5$), we could instead just replace $x$ by a single variable $x' \geq 0$ with a positivity constraint by writing $x = x' - 5$. 

Fall 2017

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