Math 340 Lecture 18

Uniqueness or non-uniqueness of solutions from the simplex method. We know that the optimal value of an LP is unique; that’s pretty much by definition. But what about the optimal solution realizing that value? There could be just one, or there could be multiple - how do we tell, and how might we find them all?

Well, the simplex method always gives us at least one optimal solution, and let’s go back to how we knew it was optimal in the first place.

\[
\begin{align*}
x_1 &= 5 & -5x_4 & +x_6 \\
x_2 &= 4 & -x_3 & +x_4 & -x_6 \\
x_5 &= 0 & -x_3 & +2x_4 & \\
z &= 13 & -3x_3 & -2x_4 & -3x_6
\end{align*}
\]

How did we know this was optimal? Because there were negative numbers for the coefficients of the objective function - the argument was that for any feasible solution \((x_1, x_2, x_3, x_4, x_5, x_6)\) we have \(z = 13 - 3x_3 - 2x_4 - 3x_6\), and since the positivity constraints force \(x_3, x_4, x_6 \geq 0\) this means 13 is definitely the maximum value \(z\) can take anywhere on the feasible region. Thinking a little harder in this case, we can see that the only way to have \(z = 13\) exactly is if \(x_3 = x_4 = x_6 = 0\). And then that forces \(x_1 = 5\), \(x_2 = 4\), \(x_5 = 0\). So \((x_1, x_2, x_3, x_4, x_5, x_6) = (5, 4, 0, 0, 0, 0)\) is the unique optimal solution (which we’d usually write as just \((x_1, x_2, x_3) = (5, 4, 0)\) and drop the slack variables).

So what if instead I wrote down

\[
\begin{align*}
x_1 &= 5 & -5x_4 & +x_6 \\
x_2 &= 4 & -x_3 & +x_4 & -x_6 \\
x_5 &= 0 & -x_3 & +2x_4 & \\
z &= 13 & -3x_3 & -3x_6
\end{align*}
\]

Well, the same argument says that we have \(z \leq 13\) always, so \((x_1, x_2, x_3) = (5, 4, 0)\) is an optimal solution attaining this optimum value. But can we have other optimal solutions? In this case yes: following our argument from above, we must keep \(x_3 = x_6 = 0\) to keep \(z = 13\), but since the coefficient for \(x_4\) is zero that can change without lowering \(z\). If we let \(x_4\) be a parameter \(t\), then our solution is

\[(x_1, x_2, x_3, x_4, x_5, x_6) = (5 - 5t, 4 + t, 0, t, 2t, 0)\]

For what values of \(t\) is this valid? Remember when we have slack variables inserted the constraints amount to saying they’re all positive, so we need to have \(5 - 5t, 2t, t \geq 0\). We can see this is for \(0 \leq t \leq 1\).

Could we find these multiple solutions by pivoting? It turns out we can. Normally we’d consider ourselves to be done at this point, but we could also say “well, I’d like to pivot around with \(x_4\) as my entering variable, which will get me to another feasible solution with \(z = 13\).” You choose the exiting variable the same as always, which is by taking the tightest constraint - in this case you can only increase \(x_4\) up to 1 until you get \(x_1\) down to zero, so you let \(x_1\) exit. You get to

\[
\begin{align*}
x_1 &= 1 & -\frac{1}{5}x_1 & +\frac{1}{5}x_6 \\
x_2 &= 5 & -x_3 & -\frac{1}{5}x_1 & -\frac{4}{5}x_6 \\
x_5 &= 2 & -x_3 & -\frac{2}{5}x_1 & +\frac{3}{5}x_6 \\
z &= 13 & -3x_3 & -3x_6
\end{align*}
\]

So we have another solution \((x_1, x_2, x_3) = (0, 5, 0)\) to go with our original one of \((5, 4, 0)\). And once we have these two optimal solutions, everything on the line segment between them is also an optimal solution (you proved this in assignment 1) - so everything of the form

\[t(0, 5, 0) + (1 - t)(5, 4, 0) = (0, 5t, 0) + (5 - 5t, 4 - 4t, 0) = (5 - 5t, 4 + t, 0)\]
for $0 \leq t \leq 1$, which is the same thing we wrote down earlier. (Depending how you wrote down the original parameter and this combination, they might not look identical like I managed to get, but they would parametrize the same set).

What would happen if I made a small change and had

\[
\begin{align*}
x_1 &= 5 + 5x_4 + x_6 \\
x_2 &= 4 - x_3 + x_4 - x_5 \\
x_5 &= 0 - x_3 + 2x_4 \\
z &= 13 - 3x_3 - 3x_6
\end{align*}
\]

Again we can set $x_4 = t$ and get $(5 + 5t, t, 0, t, 2t, 0)$ is another optimal solution. But here we just have to require $t \geq 0$, with no maximum value! And if you tried to pivot, you wouldn’t be able to find an exiting variable. In this case the set of optimal solutions is unbounded, though the original LP has an optimal solution (and is not unbounded) - the highest we can get is still $z = 13$, but we have ways of getting $z = 13$ stretching off to infinity. (We can’t conclude the linear program itself is unbounded despite having all positive coefficients for $x_4$, because you can increase $x_4$ as much as you want and it doesn’t touch the optimum value).

One more example: let’s modify our original problem in another way so we get

\[
\begin{align*}
x_1 &= 5 - 5x_4 + x_6 \\
x_2 &= 4 - x_3 + x_4 - x_5 \\
x_5 &= 0 - x_3 + 2x_4 \\
z &= 13 - 2x_4 - 3x_6
\end{align*}
\]

Now, is there more than one optimal solution? This time the answer is no, despite there being a zero in the coefficients of the objective function. Again the same argument tells you that for a solution to be optimal you must have $x_4 = x_6 = 0$, but you could potentially change $x_3$ while keeping $z = 13$. But in this case it turns out you can’t change $x_3$ at all! You have to have $x_3 \geq 0$ because of the positivity constraint, but you also have to have $x_3 \leq 0$ to keep $x_5 \geq 0$. So $x_3$ is trapped at $x_3 = 0$ and $(5, 4, 0)$ is the unique solution. In this case the 0 in the constants messed us up, even though it was harmless in the previous examples.

So summarizing, what you can say about uniqueness of optimal solutions:

- If all coefficients for $z$ are strictly negative in the optimal dictionary, the optimal solution is unique.
- If there’s a coefficient of 0 in $z$ for the optimal dictionary, and the dictionary is nondegenerate (no zeros as constants), then the optimal solution is not unique - you can always move the variable with the coefficient of 0.
- If there’s a 0 coefficient in the dictionary, but it’s also degenerate (a 0 in the coefficients)... it can go either way, you have to look more carefully at how the variables can change.

If there’s multiple optimal solutions, what does the set of them look like? In our examples, it was either a line segment (between two points) or a ray (starting at a point and going off to infinity). If there’s only one coefficient of 0, then it will always look like one of these! (Or potentially have a unique solution in a degenerate case, which we can think of as a “degenerate line segment” with both endpoints equal). If there’s multiple coefficients of 0, then things can get more complicated, and the set of optimal values may not be so easily parametrized. (In the extreme case where $z = 0$, the whole feasible region consists of optimal values!)