Math 340 Lecture 17

Finishing up from last time. We were looking at the primal LP

- Maximize $-4x_1 + 8x_2 + 7x_3$ subject to
  - $2x_1 + 2x_2 + x_3 \leq 1$
  - $x_1 - 4x_2 - 3x_3 \leq 1$
  - $x_1, x_2, x_3 \geq 0$

with dual

- Minimize $y_1 + y_2$ subject to
  - $2y_1 + y_2 \geq -4$
  - $2y_1 - 4y_2 \geq 8$
  - $y_1 - 3y_2 \geq 7$
  - $y_1, y_2 \geq 0$.

I used complementary slackness for the feasible solution $(x^*_1, x^*_2, x^*_3) = (0, 1/2, 0)$, and we were able to show it was not an optimal solution for the primal.

So okay, now I can tell you the actual optimal solution is $(x^*_1, x^*_2, x^*_3) = (0, 0, 1)$? We can do the same thing as above: the second constraint has some slack so $y^*_2 = 0$, and the third decision variable is positive so the third dual constraint must be an equality $y^*_1 - 3y^*_2 = 7$. So the dual solution we get has to be $(y^*_1, y^*_2) = (7, 0)$. Now, is this one feasible? You just check all of the constraints and find yes, it is. So we’ve proved that there exists a solution $y^*$ as in the theorem, so the theorem lets us conclude that the $x^*$ we were given was optimal for the primal, and the $y^*$ we found was optimal for the dual. And sure enough, both have value 7.

Economic interpretation of dual problems. So now I want to talk a bit about what duals mean in a real-world context. A prototypical example of a linear program in economics would be for production in, say, a factory: our variables $x_1, \ldots, x_n$ represent amounts of different goods we can produce, the objective function $c_1x_1 + \cdots + c_n x_n$ comes from good $j$ selling at a cost of $c_j$ dollars per unit, and the constraints $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$ amount to, say, limits in our production based on things like how much raw material we have, how many workers, how many machines, etc. So each $b_i$ measures a quantity of some resource, and $a_{ij}$ is the amount of resource $j$ used in producing a unit of good $i$. (In this context you’d expect we’d have $a_{ij} \geq 0$ and $b_i \geq 0$ for all $i$, though this turns out not to matter for the interpretation we’ll be talking about).

An example from the book just to give some context: say a logging company owns 100 acres with trees ready to be cut down, and $4000 in cash on hand to spend. Let’s say they have three options for any given amount of the land:

- Cut down the trees and let things regrow naturally. This yields an ultimate profit of $50 per acre but costs $10 per acre up front.
- Cut down the trees and replant it deliberately. This yields an ultimate profit of $70 per acre but costs $50 per acre up front.
- Do nothing.
Let $x_1$ be the number of acres where they cut down trees and let them regrow naturally, and $x_2$ the number of acres where they cut down trees and replant. Then our objective function is $40x_1 + 70x_2$ which we want to maximize, we have the obvious positivity constraints $x_1, x_2 \geq 0$, and we have a constraint $x_1 + x_2 \leq 100$ because there’s 100 acres available. (Notice that the slack variable for this constraint will be the “number of acres we’ll do nothing with”, so we don’t need to have a separate variable for that!) Finally we have a constraint $10x_1 + 50x_2 \leq 4000$, since we can only spend the up-front amount we have on hand. So our problem is:

- Maximize $40x_1 + 70x_2$, subject to
  - $x_1 + x_2 \leq 100$
  - $10x_1 + 50x_2 \leq 4000$
  - $x_1, x_2 \geq 0$.

If you solve this you find $x^*_1 = 25$ and $x^*_2 = 75$, which is kind of clear just from a glance - you want to use all of the wood and use the second strategy (the one that yields more profit) for as much of it as you can afford to bear the up-front cost for.

So that’s a real-world interpretation of the primal problem; but what about the dual? Here the objective function is $b_1y_1 + \cdots + b_ny_m$ and the constraints are $\sum a_{ij}y_i \geq c_j$, but what do the variables $y_i$ mean? Well, we can get a first guess just by thinking about what the units of the quantities are. In the constraint, $a_{ij}$ has units of “unit of resource $i$, per unit of good $j$” and $c_j$ is “dollars, per unit of good $j$”, so if $a_{ij} \cdot y_i$ has the same units as $c_j$ we expect $y_i$ to have units of “dollars, per unit of resource $i$”.

In our example, the dual is

- Minimize $100y_1 + 4000y_2$ subject to
  - $y_1 + 10y_2 \geq 40$
  - $y_1 + 50y_2 \geq 70$
  - $y_1, y_2 \geq 0$.

Going back to our original problem, the 100 has units of “acres”, the 4000 has units of “dollars”, the 40 and 70 are “dollars per acre”, both of the coefficients 1 are unitless (technically “acres per acre”), and the 10 and 50 are “dollars per acre”. So looking at the constraints for the dual, we see that $y_1$ should be “dollars per acre”, $y_2$ should be “dollars later per dollar now”), and the units of what we’re trying to minimize is again dollars (which makes sense - duality compares it to the objective function of the primal which is definitely also in dollars).

But what does this dual problem actually mean? We’ve put units on it but it still looks kind of like gibberish. But looking at the units, you may start to suspect $y_i$ is the marginal cost of resource $i$: how much you should be willing to spend to buy another unit of resource $i$. This sort of dimensional analysis suggests that, but of course isn’t a proof (or even a precise claim as to what exactly “marginal cost” means in the context of this linear programming problem). But fortunately the following theorem says that it is actually true.

**Marginal cost theorem.** A qualitative statement of the marginal cost theorem is: Let $\mathbf{x}^*$ and $\mathbf{y}^*$ be (nondegenerate) optimal solutions for the primal and the dual. If you increase any of the constants $b_i$ in the primal by a small amount (denoted $\Delta b_i$), then the optimum value increases by $(\Delta b_i)y_i$. Flipping this around: if you can buy more resources to increase $b_i$ for a rate less than $y_i$, then you can profit by doing so! (But key point here is “a small amount” - after some point it may stop increasing the objective value at that rate).

Let’s go back to our example. What are $y^*_1$ and $y^*_2$? Well, we could get them by the simplex method, but we could also use complementary slackness like we talked about at the beginning of the lecture - since $x^*_1, x^*_2 > 0$, both constraints for the dual must have no slack. So $y^*_1 + 10y^*_2 = 40$ and $y^*_1 + 50y^*_2 = 70$, and
we can solve these two equations (subtract one from the other to get $40y_2 = 30$ and thus $y_2 = .75$, and then $y_1 = 32.5$).

So $y_1$ is 32.5 dollars per acre, and the theorem tells us that this is the rate at which we should be willing to buy extra land with trees to increase the constant $100$ in the first upper bound. Why shouldn’t we be willing to buy at closer to $40$ dollars per acre, since that’s how much profit we can make from them by using the first strategy? Well, because even that has an up-front cost, which reduces the amount of land we can invest in the more profitable second strategy for.

Also, $y_2$ is $.75$ dollars (of up-front cost) now per dollar (of profit) later. What this means is it we would break even if we took a loan where for every one dollar we got now, we paid back $1.75$ after reaping all of the profit from our strategy. So if we had an option to take a $100$ loan and pay back $150$ after all is said and done (a net decrease of $50$ from our ultimate balance), we would want to do that because that $100$ loan would result in a $75$ dollar increase in our profit.

We’ll talk about a more quantitative version of this later - you can pin down the range where this ends up being valid (it really amounts to “until you run into another constraint”, e.g. it would stop being worth buying more land at 32.5 dollars per acre when you got to the point that you didn’t have enough initial money to do either strategy on all of it, or it would stop being worth taking loans once you had enough to do strategy 2 on everything. The “this is valid for small increases” should make a lot of intuitive sense - for most real-world problems you could expect that your marginal cost would change if you start going far away from your initial conditions.