Math 340 Lecture 13

**Formal statement of weak duality.** Last time we introduced the dual to an LP (in standard form), which was most succinctly given in terms of matrix notation: if the primal problem was

- Maximize $\vec{c} \cdot \vec{x}$, subject to $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$,

the dual LP is

- Minimize $\vec{b} \cdot \vec{y}$, subject to $A^\top \vec{y} \geq \vec{c}$ and $\vec{y} \geq 0$

(which we again have to put into standard form if we want to do the simplex method, but that ends up being straightforward). The justification I gave for introducing the dual was to give upper bounds for the primal, which is summed up formally in the following theorem:

**Weak duality theorem:** Suppose we have a primal and dual LP as above. If $\vec{x}$ is any feasible solution to the primal and $\vec{y}$ is any feasible solution to the dual, then $\vec{c} \cdot \vec{x} \leq \vec{b} \cdot \vec{y}$ (these are the values of the objective functions for the two problems at the two solutions)!

Corollary: The optimal value of the primal is $\leq$ the optimal value of the dual. (If both problems actually have optimal values!)

The fact that I’m calling this the “weak” duality theorem probably suggests that there’s going to be a “strong” duality theorem too. That’s indeed true - the strong duality theorem is that if I take what I wrote down as the corollary, the optimal values are actually equal. The statements of weak and strong duality look similar, but they are asking for fairly different things: weak duality just compares any pair of feasible solutions I have on hand and gives an inequality, while strong duality is basically asking me to produce two feasible solutions here we have an equality. So the proof of strong duality (which I’ll give on Friday) is necessarily harder.

**Proof of weak duality:** Well, one proof is “we did it in class last time”! The whole way we set up the dual problem was to make the inequality I wrote down true. But it’s good to think about these sort of things from a few different angles, so I want to give another proof using the matrix notation.

The proof is again pretty simple: we just need to get a chain of inequalities, using (all of!) the parts of the matrix definition of the two LPs. The chain is just

$$\vec{c} \cdot \vec{x} \leq (A^\top \vec{y}) \cdot \vec{x} = \vec{y} \cdot (A\vec{x}) \leq \vec{y} \cdot \vec{b}.$$ 

The first $\leq$ is true because $\vec{c} \leq A^\top \vec{y}$ and $\vec{x} \geq 0$ (we need a positivity condition to multiply a single inequality by a scalar, and adding this all up we need it for all of the variables to get something for the dot product). Similarly the last $\leq$ is true because $A\vec{x} \leq \vec{b}$ and $\vec{y} \geq 0$. Why is the middle inequality true? This goes back to linear algebra - one way to phrase it is that $A$ and $A^\top$ are a pair of adjoint matrices, which means exactly you can shift one across the dot product by replacing it with the other. And the proof of this is actually really easy and comes back to remembering that dot products are just matrix multiplication by the transpose:

$$(A^\top \vec{y}) \cdot \vec{x} = (A^\top \vec{y})^\top \vec{x} = (\vec{y}^\top A^\top) \vec{x} = \vec{y}^\top (A\vec{x}) = \vec{y} \cdot (A\vec{x}).$$

**Possibilities for a primal-dual pairs of LPs.** The fundamental theorem of linear programming tells us that we have three possibilities for each LP: either it’s infeasible, unbounded, or has an optimal value. So for the primal and dual together, that’s hypothetically 9 possibilities. Which of them can actually happen? Well, weak duality lets us rule out something right away:

- If our primal LP is unbounded, then the dual must be infeasible.

Why is this? Well, weak duality says any feasible solution of the dual provides an upper bound on the primal. So if the primal is unbounded, that can’t happen -there can’t be any feasible solutions to the dual. We also have the statement with “primal” and “dual” reversed.

- If our dual LP is unbounded, then the primal must be infeasible.
So that rules out quite a few possibilities. We can then make a table of which things are possible:

<table>
<thead>
<tr>
<th>Primal</th>
<th>Infeasible</th>
<th>Optimal</th>
<th>Unbounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infeasible</td>
<td>(Yes)</td>
<td>(No)</td>
<td>Yes</td>
</tr>
<tr>
<td>Dual Optimal</td>
<td>(No)</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Unbounded</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

So we’ve ruled out three possibilities via weak duality, and we have a diagonal of three “yes” situations that are what we sort of expect: either both the primal and dual have an optimal solution, or one is unbounded and the other is infeasible. This leaves three other entries in the grid to consider. There are two I’ve written in “no” on parentheses - these are because we need strong duality to justify these “no’s. And there’s one “yes” in parentheses, which is possible but is kind of uninteresting; if both the primal and the dual are infeasible then there’s not so much to really say.

The theorem of the alternative. Now we want to use linear programming to prove something about linear inequalities (but which isn’t really about linear programming itself).

**Theorem of the alternative.** Let $A$ be an $m \times n$ matrix and $\vec{b}$ a vector in $\mathbb{R}^m$. Then exactly one of the following two things is true:

- There exists a vector $\vec{x} \in \mathbb{R}^n$ with $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$.
- There exists a vector $\vec{y} \in \mathbb{R}^m$ with $A^\top \vec{y} \geq 0$, $\vec{y} \geq 0$, and $\vec{b} \cdot \vec{y} < 0$.

So this isn’t directly about linear programming, though we can certainly see how it might be connected. And in any case, it’s kind of a neat abstract statement, that we can neatly divide things into two boxes. (Or, if you’re looking for a vector $\vec{x}$ satisfying the constraints $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$, it tells you that the failure of this can be certified by finding some other $\vec{y}$ satisfying a certain thing).

So how do we prove this? You can probably guess I want to set up a linear program and its dual, and use the table I wrote above to pin us down between two situations. And the primal LP should have constraints $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$, which is just like normal. But what should the objective function be? Well, it will be of the form “maximize $\vec{c} \cdot \vec{x}$, where $\vec{c}$ will show up in the dual program in the constraint $A^\top \vec{y} \geq \vec{c}$. So looking at what we’ll want the dual to be, we take $\vec{c} = 0$, i.e. the objective function is zero! So our primal and dual are:

- Maximize $0 = 0 \cdot \vec{x}$ subject to $A\vec{x} \leq \vec{b}$ and $\vec{x} \geq 0$.
- Minimize $\vec{b} \cdot \vec{y}$ subject to $A^\top \vec{y} \geq 0$ and $\vec{y} \geq 0$.

Looking at this, the primal is either infeasible or has an optimal value (if it has any feasible solutions they’re all optimal because the value of $z$ is always zero), while the dual either has an optimal solution or is unbounded (it’s certainly feasible because $\vec{y} = 0$ is a feasible solution with value 0). So now we look at our chart for which combinations are possible:

- The primal is optimal and the dual is optimal is **possible**. In this case the first statement of our “alternative” is true, and the second is false (weak duality provides a lower bound of 0 on $\vec{b} \cdot \vec{y}$).
- The primal is infeasible and the dual is unbounded is **possible**. In this case the second statement of our “alternative” is true, and the first is false: an unbounded minimization problem means we can find a feasible solution $\vec{y}$ with objective value $\vec{b} \cdot \vec{y}$ as small as we want, so certainly $< 0$.
- The primal is optimal and the dual is unbounded is **not possible**, by weak duality. This situation would have corresponded to both statements of the alternative being true.
- The primal is infeasible and the dual is optimal is **not possible**, by strong duality. This would have corresponded to neither statements of the alternative being true.

So looking at our table, the situations where exactly one of the two options is true are both possible, and the situations where both or neither are true are impossible. This proves the theorem!