Problem 1. For this problem and the next two, consider the following linear program and the given optimal solution.

- Maximize \( \vec{c} \cdot \vec{x} \) subject to \( A \vec{x} \leq \vec{b} \) and \( \vec{x} \geq 0 \), where:

\[
A = \begin{bmatrix}
1 & 0 & -1 & 2 \\
2 & 1 & 1 & 0 \\
3 & 1 & -2 & 1
\end{bmatrix}, \quad 
\vec{b} = \begin{bmatrix}
1 \\
2 \\
2
\end{bmatrix}, \quad 
\vec{c} = \begin{bmatrix}
1 \\
2 \\
-2 \\
3
\end{bmatrix}.
\]

(a) The optimal solution for this LP occurs with

\[
\vec{x}_B = \begin{bmatrix}
x_2 \\
x_3 \\
x_4
\end{bmatrix}, \quad 
\vec{x}_N = \begin{bmatrix}
x_1 \\
x_5 \\
x_6 \\
x_7
\end{bmatrix}, \quad 
\vec{b}^* = \begin{bmatrix}
9/5 \\
1/5 \\
3/5
\end{bmatrix}.
\]

Write down the matrices \( A_B \) and \( A_N \) and the vectors \( \vec{c}_B, \vec{c}_N \) associated with this dictionary.

(b) Compute the objective function (including the optimal value!) for the dictionary specified in part (a).

Problem 2. In this problem and the next two, you’ll do some sensitivity analysis for the LP from Problem 1. (The two parts of this question are independent of each other - the modifications are considered separately).

(a) Suppose we want to consider modifying the coefficient of \( x_1 \) in the objective function, so we replace it with a variable:

\[
\vec{c} = \begin{bmatrix}
c_1 \\
2 \\
-2 \\
3
\end{bmatrix}.
\]

For what range of values of \( c_1 \) does our original dictionary with \( \vec{x}_B \) stay optimal, and what is the objective value in this range?

(b) Consider instead modifying the third constant in our system of equations:

\[
\vec{b} = \begin{bmatrix}
1 \\
2 \\
b_3
\end{bmatrix}.
\]

For what range of values of \( b_3 \) does our original dictionary with \( \vec{x}_B \) stay optimal, and what is the objective value in this range?

Problem 3. Once again, use the LP from Problem 1.

(a) Now consider modifying the coefficient of \( x_4 \) in the objective function instead:

\[
\vec{c} = \begin{bmatrix}
1 \\
2 \\
-2 \\
c_4
\end{bmatrix}.
\]

For what range of values of \( c_4 \) does our original dictionary with \( \vec{x}_B \) stay optimal?

(b) Suppose that in the case of part (a), \( c_4 \) is increased to 10 (which should be outside of the range you found). Make a pivot from the dictionary defined by \( \vec{x}_B \) to find a new optimal solution and its optimum value. (Make sure to check that your new dictionary is actually optimal!)
Problem 4. Prove the following version of the theorem of the alternative, that for any $m \times n$ matrix $A$ and vector $\vec{b} \in \mathbb{R}^m$ exactly one of the following is true:

1. There exists a vector $\vec{x}$ in $\mathbb{R}^n$ satisfying $0 \leq A\vec{x} \leq \vec{b}$.

2. There exists two vectors $\vec{y}_1, \vec{y}_2$ in $\mathbb{R}^m$ satisfying $A^T(\vec{y}_1 - \vec{y}_2) = 0$, $\vec{y}_1, \vec{y}_2 \geq 0$, and $\vec{b} \cdot \vec{y}_1 < 0$.

(If you’re having trouble getting started, make sure to set up an LP with the constraints specified in problem (1) and put it in a form you know how to take the dual of!)