LECTURE 7 - 1/24/17

- YOU SHOULD BE ENROLLED IN THIS SECTION OFFICIALLY NOW! TALK TO ME IF NOT.
- 1ST MIDTERM: IN CLASS, TUES JAN 31. (50 MINUTES)

  COVER 1.1 - 1.9.

  PRACTICE PROBLEMS & EXAMS ON WEBSITE.

LINEAR TRANSFORMATIONS: GIVEN AN $m \times n$ MATRIX A, DEFINE THE MATRIX TRANSFORMATION

$$T: \mathbb{R}^n \to \mathbb{R}^m \qquad T(v) = Av$$

EXAMPLE: $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 520 & 390 & 290 \\ 28 & 18 & 13 \\ 24 & 17 & 17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 \begin{bmatrix} 520 \\ 28 \\ 24 \end{bmatrix} + x_2 \begin{bmatrix} 390 \\ 18 \\ 17 \end{bmatrix} + x_3 \begin{bmatrix} 290 \\ 13 \\ 17 \end{bmatrix}.$$

EXAMPLE INTERPRETATION: x_1 = # HAMBURGERS

x_2 = # PIZZA SLICES

INPUT $\rightarrow$ x_3 = # TACOS

OUTPUT $\longrightarrow$ $\begin{bmatrix} \ \\ \ \\ \ \end{bmatrix}$ = CALORIES / GRAMS FAT / GRAMS PROTEIN.

WANT TO THINK ABSTRACTLY ABOUT HOW T BEHAVES. IT SATISFIES:

$$\left.\begin{array}{l} T(2x) = 2\cdot T(x). \\ T(ax) = a\cdot T(x). \\ T(x+y) = T(x)+T(y). \end{array}\right]$$ TRANSFORMATION RULES.

DEFINITION: A LINEAR TRANSFORMATION IS ANY $T: \mathbb{R}^n \to \mathbb{R}^m$ SATISFYING

$$\left[\begin{array}{l} T(x+y) = T(x)+T(y) \\ T(ax) = a\cdot T(x) \end{array}\right.$$
FOR ANY x, y IN $\mathbb{R}^n$.
FOR ANY x IN $\mathbb{R}^n$ AND ANY SCALAR a.

NOTE: "LINEAR TRANSFORMATION" IS AN ABSTRACT DEFINITION, VS. "MATRIX TRANSFORMATION" IS EXPLICIT.

BIG THEOREM: MATRIX TRANSFORMATIONS AND LINEAR TRANSFORMATIONS ARE THE SAME THING!!!

2 PARTS TO THIS THEOREM: $\boxed{T(v) = Av}$

① MATRIX TRANSFORMATIONS ARE LINEAR

THIS IS JUST ALGEBRA OF MATRIX MULTIPLICATION

$$T(v+w) = A(v+w) \overset{?}{=} Av + Aw = T(v) + T(w)$$
TRUE!

SIMILARLY $T(av) = a\,T(v)$

2 LINEAR TRANSFORMATIONS ARE MATRIX TRANSFORMATIONS

GIVEN ANY SORT OF $T: \mathbb{R}^n \to \mathbb{R}^m$, AS LONG AS IT SATISFIES OUR RULES, IT MUST COME FROM A MATRIX: WE CAN FIND A WITH

$$T(v) = Av.$$

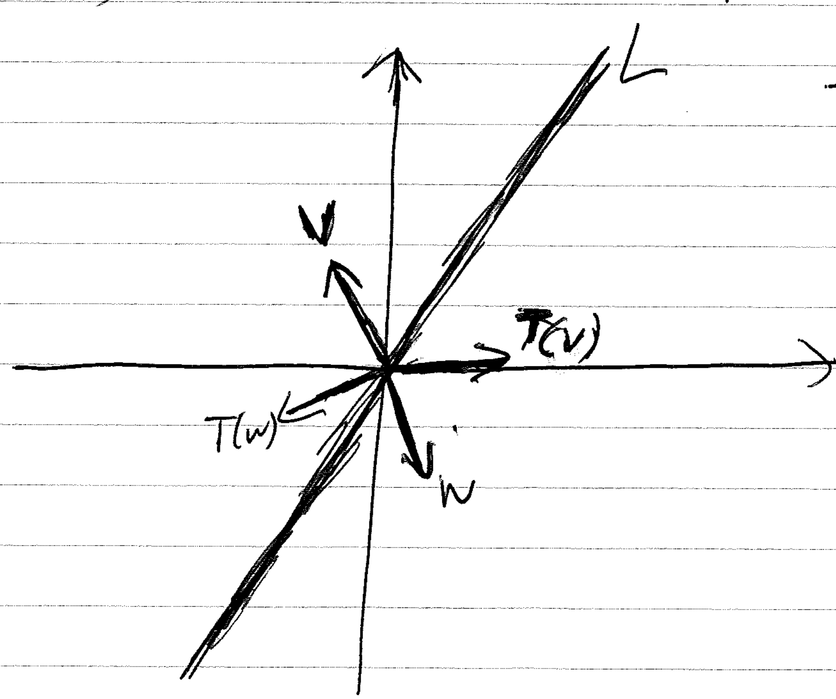
WHAT SORT OF LINEAR TRANSFORMATIONS T CAN WE DEFINE WITHOUT TALKING ABOUT MATRICES?

GEOMETRIC LINEAR TRANSFORMATIONS

$$T: \mathbb{R}^2 \to \mathbb{R}^2.$$

EXAMPLE: REFLECTIONS

IDEA: HAVE A LINE L THROUGH ORIGIN. T REFLECTS VECTORS ACROSS THAT LINE.



T DEFINED BY:

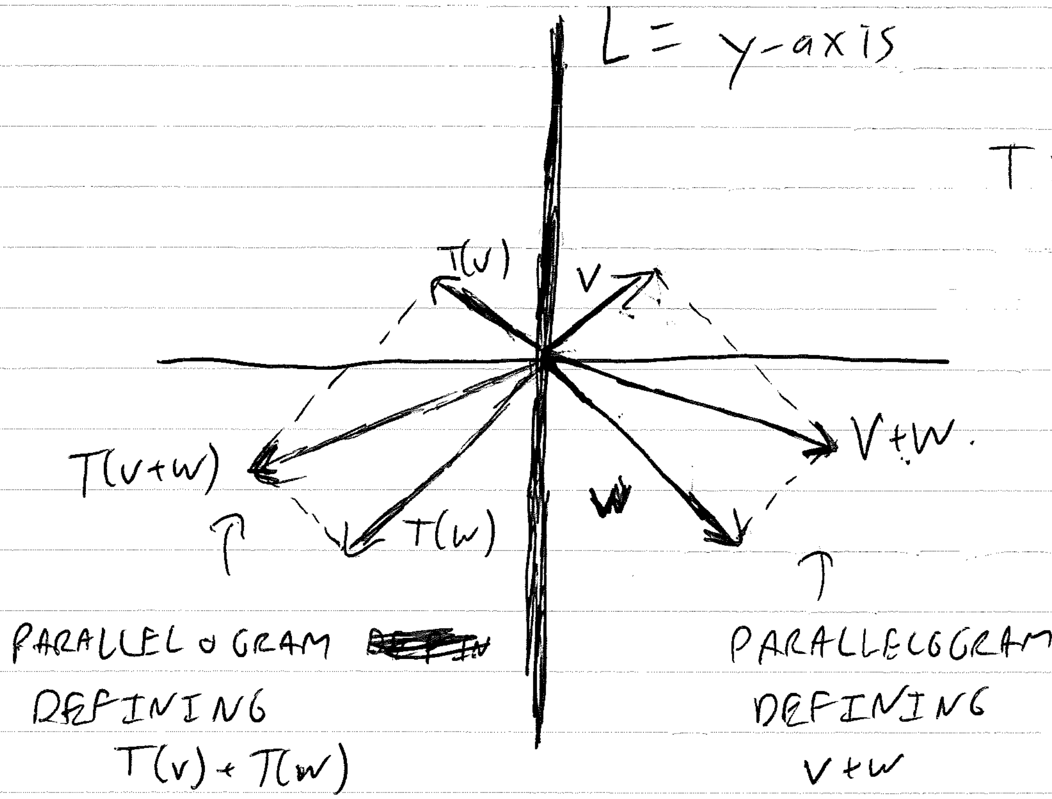
INPUT VECTOR v,
OUTPUT $T(v)$
REFLECTED ACROSS LINE L.

WHY IS T COMING FROM REFLECTION LINEAR?
NEED

$$T(v+w) = T(v) + T(w).$$

REMEMBER: ADDITION COMES FROM PARALLELOGRAM RULE GEOMETRICALLY.

T: REFLECTS ACROSS Y-AXIS.



SO $\underline{T(v+w) = T(v) + T(w)}$

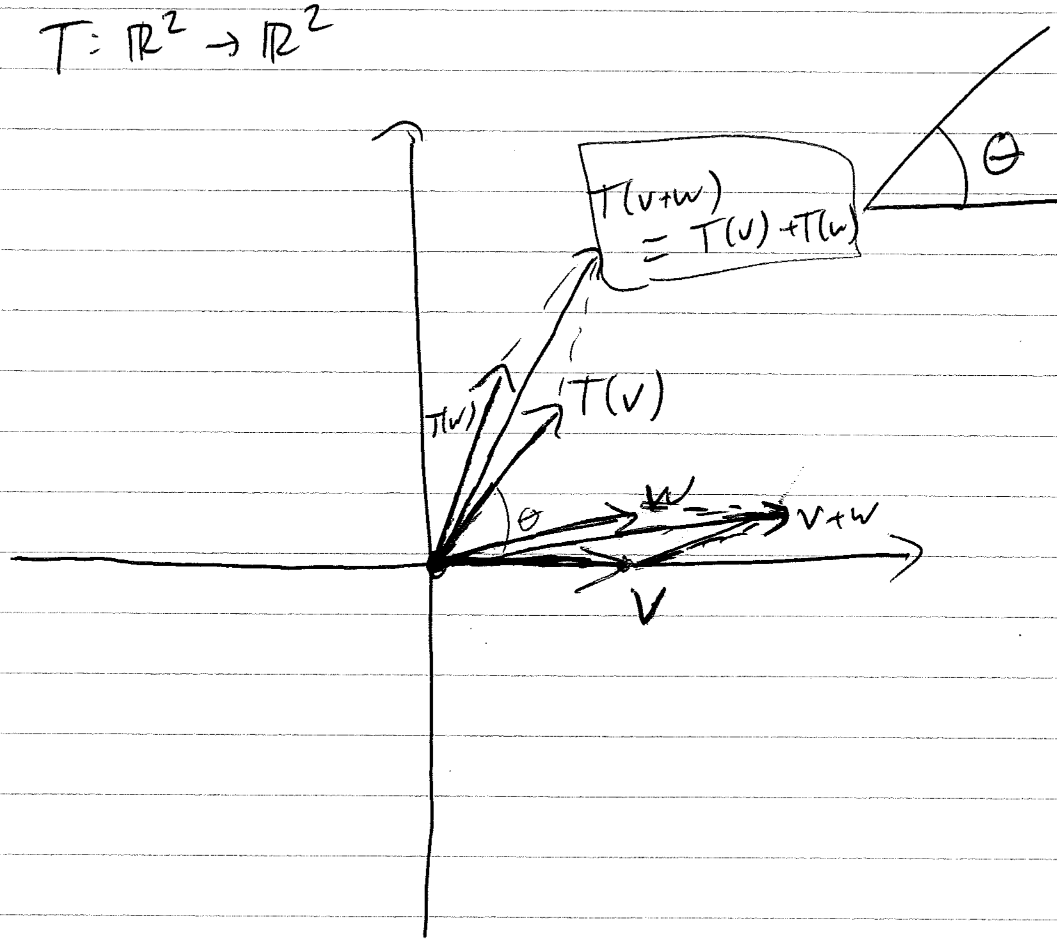
SIMILARLY: $T(av) = a \cdot T(v)$.

SO THIS T IS A LINEAR TRANSFORMATION.
$\Rightarrow$ THERE IS A MATRIX A (2×2) WITH $T(v) = Av$

WHAT IS IT? WILL COME BACK TO THAT!

EXAMPLE 2: T COMES FROM ROTATION ABOUT THE ORIGIN BY AN ANGLE θ.

$T: \mathbb{R}^2 \to \mathbb{R}^2$



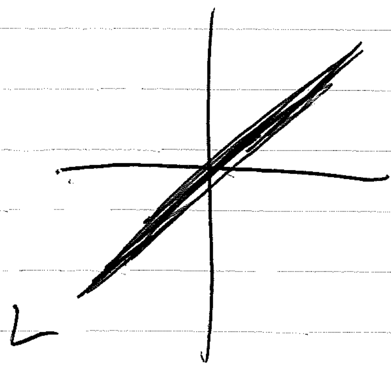
SO THIS ROTATION IS A LINEAR TRANSFORMATION.

WHAT ARE THE MATRICES A THAT GIVE THESE GEOMETRIC LINEAR TRANSFORMATIONS?

- T = REFLECTION ABOUT y-axis.

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x + 0 \\ 0 + y \end{bmatrix}.$$

- T = REFLECTION ABOUT LINE $y=x$.



$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

- T = SOME WEIRD LINE L ???

- T = ROTATION ABOUT ORIGIN BY θ

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

HOW DO I FIND THIS MATRIX?
THINK ABOUT WHAT LINEARITY TELLS US:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

$$\boxed{= x\,T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + y\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).}$$

SO: IF A HAS $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ AS 1ST COLUMN,

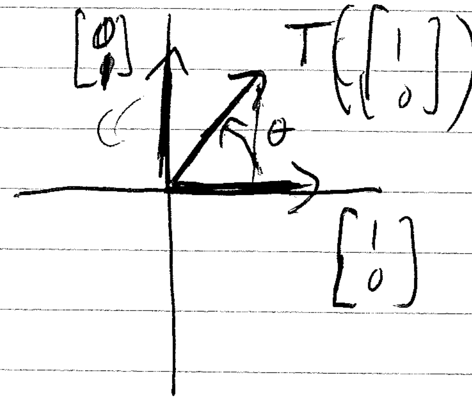
$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ AS 2ND COLUMN,

$T(v) = Av$.

WHAT IS $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$?

ROTATIONS ALWAYS AROUND ORIGIN.

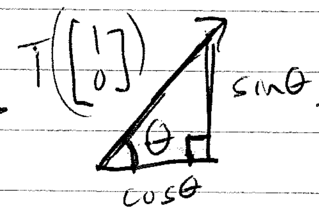
T = ROTATE BY θ.



REMEMBER SOME TRIGONOMETRY...

$\begin{bmatrix}1\\0\end{bmatrix}$ ROTATED BY ANGLE θ

$\rightsquigarrow \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$.



SIMILARLY: $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$.

SO: $\boxed{A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}}$ IS MATRIX FOR ROTATION BY θ.

$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ ALWAYS OUTPUTS "$\begin{bmatrix}x\\y\end{bmatrix}$ ROTATED BY ANGLE θ".

$\theta = 90^\circ$: $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-y\\x\end{bmatrix}$