LECTURE 6 - 1/19/17

QUIZ 1 GRADED - PICK UP AT MATH LEARNING CENTER
(LSK 301/302)

LINEAR INDEPENDENCE

$$\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$$

LINEARLY DEPENDENT BECAUSE ONE VECTOR A LINEAR COMBINATION OF THE OTHERS.

LIST IS "REDUNDANT" BECAUSE WE CAN THROW OUT A VECTOR BUT STILL KEEP SAME SPAN

$$\text{SPAN}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right) = \text{SPAN}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right).$$

FORMAL DEFINITION:

GIVEN A LIST OF VECTORS $v_1, \ldots, v_n$ IN $\mathbb{R}^m$, A DEPENDENCE RELATION IS ANY LINEAR COMBINATION SUMMING TO 0:

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = 0.$$

THE TRIVIAL DEPENDENCE RELATION IS

$$0 v_1 + 0 v_2 + \cdots + 0 v_n = 0.$$

ALL SCALARS ARE ZERO.

THE LIST OF VECTORS $v_1, \ldots, v_n$ IS:

- **LINEARLY DEPENDENT** IF THERE IS A NONTRIVIAL DEPENDENCE RELATION
- **LINEARLY INDEPENDENT** IF THERE IS ONLY THE TRIVIAL DEPENDENCE RELATION.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ IS LINEARLY **DEPENDENT**.

$$1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

NONTRIVIAL DEPENDENCE RELATION.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ IS LINEARLY **INDEPENDENT**.

NEED TO SHOW **NO** NONTRIVIAL DEPENENCE RELATIONS,

I.E. ONLY SOLUTION TO

$$x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

IS $x_1 = x_2 = 0$.

IF $x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

SO $x_2 = 0$ SO $x_1 = 0$.

SO JUSTIFIED **ONLY** THE TRIVIAL DEPENDENCE RELATION, SO LINEARLY **INDEPENDENT**.

SO LINEAR DEPENDENCE TELLS US ~~WE~~ CAN THROW SOMETHING OUT.

EXAMPLE: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix},$ LINEARLY DEPENDENT

CAN THROW OUT EITHER $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}$

BUT NOT $\begin{bmatrix}1\\1\end{bmatrix}$

NEXT EXAMPLE:

$$v_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}1\\-2\\1\end{bmatrix}, \quad v_3 = \begin{bmatrix}3\\-2\\-1\end{bmatrix} \quad v_4 = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$\boxed{\text{LINEARLY DEPENDENT}}$ OR LINEARLY INDEPENDENT?

NOTE: 4 VECTORS IN $\mathbb{R}^3$.

FINDING DEPENDENCE RELATIONS
= SOLVING A HOMOGENEOUS SYSTEM
$x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = 0.$

4 VARIABLES, 3 EQUATIONS.
$\Rightarrow$ ALWAYS A FREE VARIABLE,
SO ALWAYS A NONTRIVIAL SOLUTION
= A NONTRIVIAL DEPENDENCE RELATION

TO FIND A DEPENDENCE RELATION, SOLVE.

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ -1 & -2 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

ROW REDUCE:

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 0 \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} \underline{1} & 1 & 3 & 1 & 0 \\ 0 & \underline{-1} & 1 & 2 & 0 \\ 0 & 0 & 0 & \underline{3} & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$x_1 = -4x_3$
$x_2 = x_3$
x_3 FREE
$x_4 = 0$

DEPENDENCE RELATIONS ARE.

$$-4x_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$x_3 = 1$ GIVES A SPECIFIC NONTRIVIAL DEPENDENCE RELATION.

$$-4 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

WE CAN SEE:

- DEPENDENCE RELATIONS ON $v_1, v_2, \ldots, v_n$.
  = SOLUTIONS OF THE HOMOGENEOUS SYSTEM
  $Ax = 0$, $A = [v_1, \ldots, v_n]$

SO GIVEN A LIST OF SPECIFIC VECTORS,
SOLVE $Ax = 0$: NO FREE VARIABLES = LINEARLY INDEPENDENT
FREE VARIABLES = LINEARLY DEPENDENT.

WOULD LIKE: SOME RULES TELLING US RIGHT AWAY OUR LIST IS DEPENDENT/INDEPENDENT IN "EASY" SITUATIONS.

1. IF THE LIST OF VECTORS $v_1, \ldots, v_n$ INCLUDES THE ZERO VECTOR 0, IT'S LINEARLY DEPENDENT.

   $$\underline{0} \cdot v_1 \cdot - \boxed{1} \cdot 0 \quad \cdot \quad - \underline{0} \cdot v_n = 0.$$

2. IF YOUR LIST CONTAINS TWO VECTORS THAT ARE SCALAR MULTIPLES OF EACH OTHER, IT'S LINEARLY DEPENDENT.

3. IF YOU CAN SEE ONE VECTOR IN YOUR LIST IS A LINEAR COMBINATION OF OTHERS, IT'S LINEARLY DEPENDENT.

4. IF YOU HAVE $v_1, \ldots, v_n$ IN $\mathbb{R}^m$ WITH $n > m$, IT'S ALWAYS LINEARLY DEPENDENT.

WHAT ABOUT RULES GUARANTEEING LINEAR INDEPENDENCE?
A BIT TRICKIER:

1. FOR JUST ONE VECTOR v_1, THE LIST $\{v_1\}$ IS LINEARLY INDEPENDENT AS LONG AS $v_1 \neq 0$.

   (DEPENDENCE RELATION: $x_1 v_1 = 0$)

2. FOR TWO VECTORS v_1, v_2 THE LIST v_1, v_2 IS LINEARLY INDEPENDENT IF $v_1 \neq 0$, AND v_2 IS NOT A SCALAR MULTIPLE OF v_1.

3. FOR THREE VECTORS v_1, v_2, v_3, THE LIST IS LINEARLY INDEPENDENT IF v_1, v_2 ARE LINEARLY INDEPENDENT, AND v_3 IS NOT A LINEAR COMBINATION OF v_1 & v_2.

4. RECURSIVELY: A LIST $v_1, \dots, v_n$ IS LINEARLY INDEPENDENT IF THE LIST $v_1, \dots, v_{n-1}$ IS LINEARLY INDEPENDENT AND v_n IS NOT A LINEAR COMBINATION OF $v_1, \dots, v_{n-1}$.

~~EXAMPLE~~:

NOTE: CHANGING ORDER OF VECTORS DOES NOT AFFECT LINEAR DEPENDENCE/INDEPENDENCE.

EXAMPLE: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ LINEARLY INDEPENDENT.

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ LINEARLY INDEPENDENT.

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ LINEARLY INDEPENDENT (3RD IS NOT LIN. COMB. OF 1ST & 2ND)

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$ LINEARLY <u>DEPENDENT</u> (3RD IS A LINEAR COMBINATION OF 1ST & 2ND)

$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ LINEARLY <u>DEPENDENT</u>

(EVEN THOUGH 4TH VECTOR ISN'T LINEAR COMBINATION OF FIRST THREE, BECAUSE THE FIRST 3 AREN'T LINEARLY INDEPENDENT)

$$-4\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A FEW CLASSES AGO: I SAID SPAN(v_1, v_2) IS "USUALLY" A PLANE ($v_1, v_2 \in \mathbb{R}^3$).

WHAT I REALLY MEAN: AS LONG AS v_1, v_2 ARE ARE LINEARLY INDEPENDENT, SPAN(v_1, v_2) IS A PLANE.

WHEN WE HAVE LINEAR INDEPENDENCE, 2 VECTORS SPAN A 2-DIMENSIONAL SPACE.

LINEAR TRANSFORMATIONS: WANT TO THINK ABOUT MATRICES AS GIVING US FUNCTIONS.

GIVEN AN $m \times n$ MATRIX A, IT WILL GIVE US A FUNCTION $T: \mathbb{R}^n \to \mathbb{R}^m$

"INPUT"

WHAT THIS MEANS: FOR EVERY VECTOR v IN $\mathbb{R}^n$ (THE DOMAIN) WE CAN PUT IT INTO THE FUNCTION AND GET A VECTOR $T(v)$ IN $\mathbb{R}^m$. (THE CODOMAIN).

"OUTPUT".

(NOTE: RANGE IS SUBSET OF CODOMAIN THAT'S ACTUALLY HIT BY VALUES $T(v)$)

GIVEN A ($m \times n$), WE DEFINE T BY THE FORMULA $\boxed{T(v) = Av}$

(MATRIX TRANSFORMATION)

EX: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ 3×2

$T : \mathbb{R}^2 \to \mathbb{R}^3$

$$\boxed{T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{bmatrix}}$$

THINKING ABOUT ~~MATRIX A~~ T
AS TRANSFORMING DATA $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ BY A RULE.

SILLY EXAMPLE:

$$T : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{matrix} \text{HAMBURGERS} \\ \text{PIZZA SLICES} \\ \text{TACOS} \end{matrix} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \begin{matrix} \text{CALORIES} \\ \text{GRAMS FAT} \\ \text{GRAM PROTEIN} \end{matrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_1 \begin{bmatrix} 520 \\ 28 \\ 24 \end{bmatrix} + x_2 \begin{bmatrix} 390 \\ 18 \\ 17 \end{bmatrix} + x_3 \begin{bmatrix} 290 \\ 13 \\ 17 \end{bmatrix}$$

DATA FOR SINGLE HAMBURGER — SINGLE PIZZA SLICE — SINGLE TACO.

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 520 & 390 & 290 \\ 28 & 18 & 13 \\ 24 & 17 & 17 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

MATRIX TRANSFORMATION.